Screen Integrable Lightlike Hypersurfaces of Indefinite Sasakian Manifolds

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Abstract. We investigate lightlike hypersurfaces of indefinite Sasakian manifolds, tangent to the structure vector field ξ and whose screen distribution is integrable. We prove some results on parallel vector fields and on a leaf of the integrable distribution $D_0 \perp \langle\xi\rangle$ of this class. A theorem on a geometrical configuration of the screen distribution is obtained. We show that any totally contact umbilical leaf of a screen integrable distribution of a lightlike hypersurface is an extrinsic sphere.

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1. Introduction

The general theory of degenerate submanifolds of semi-Riemannian (or Riemannian) manifolds is one of the interesting topics of differential geometry. Since for any semi-Riemannian manifold there is a natural existence of lightlike subspaces, their study is equally desirable, but, due to the degenerate induced metric of a lightlike submanifold, one fails to use the theory of non-degenerate geometry in the usual way. The primary difference between the lightlike submanifolds and the non-degenerate submanifolds is that in the first case the normal vector bundle intersects the tangent bundle. Thus, the study becomes more difficult and strikingly different from the study of non-degenerate submanifolds. To deal with this anomaly, the lightlike submanifolds were introduced and presented in a book by Duggal and Bejancu [9]. They introduced a non-degenerate screen distribution to construct a nonintersecting lightlike transversal vector bundle of the tangent bundle. Unfortunately, the induced objects on a lightlike submanifold depend on the choice of a screen distribution which, in general, is not unique. For a submanifold of an indefinite Sasakian manifold, some aspects have been studied in [3] and its

lightlike case is very limited and some discussions can be found in [5], [11], [13] and [14].
Physically, lightlike hypersurfaces are interesting in general relativity since they produce models of different types of horizons. For instance, the existence of Killing vector fields has been often used as the most effective symmetry. In fact, since the Einstein's field equations are a complicated set of nonlinear partial differential equations, many exact solutions have been found by assuming one or more Killing vector fields (see [8] and [9] for more details and many more references therein). In particular, Carter [6] used this information in the study of a null (lightlike) hypersuface which is also a Killing horizon. Lightlike hypersurfaces are also studied in the theory of electromagnetism (see, for instance [9, Chapter 8]).
Duggal and Bejancu discuss the Cauchy–Riemann (CR) lightlike submanifolds of indefinite Kaehler manifolds in [9, Chapter 6] and prove that, in a totally umbilical real lightlike hypersurface of an indefinite Kaehler space form, the nonzero mean curvature vector satisfies partial differential equations which imply that the nonzero mean curvature vector is not parallel. The usual terminology says that such an umbilical lightlike submanifold is not an *extrinsic sphere* (see [7] for more details and many more references therein). As the notion of totally umbilical submanifolds of Kaehlerian manifolds corresponds to that of totally contact umbilical submanifolds of Sasakian manifolds [12], the author in [13] showed that, in a totally contact umbilical lightlike hypersurface of an indefinite Sasakian space form, the nonzero mean curvature vector also is not parallel.
In the present paper, we study the geometry of lightlike hypersurfaces of indefinite Sasakian manifolds, tangent to the structure vector field, by particularly paying attention to the geometry of screen integrable lightlike hypersurfaces. The paper is organized as follows. In Section 2, we recall some basic definitions for indefinite Sasakian manifolds and lightlike hypersurfaces of semi-Riemannian manifolds. In Section 3, we give the decomposition of almost contact metrics of lightlike hypersurfaces in indefinite Sasakian manifolds which are tangential to the structure vector field as well as theorems on Lie derivatives. In Section 4, some theorems on parallel vector fields and integrability of the distribution $D_0 \perp \langle\xi\rangle$ (Theorems 4.3, 4.4 and 4.7) are stated. We prove that, if any leaf of the integrable distribution $D_0 \perp \langle\xi\rangle$ is totally geodesic, then $\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))$ is a Killing distribution. Moreover, if it is parallel, then $\overline{\phi}(TM^{\perp})$ and $\overline{\phi}(N(TM))$ are Killing distributions on that leaf (Theorem 4.8). By Theorem 5.3 in Section 5, we establish the geometrical configuration of the screen distributions of a lightlike hypersurface in Sasakian space forms. We also show that any totally contact umbilical leaf of an integrable screen distribution of a lightlike hypersurface is an extrinsic sphere (Theorem 5.5). Finally, in Section 6 we discuss the effect of any change of the screen distribution on some different found results.

2. Preliminaries

Let $\overline{M}$ be a $(2n+1)$-dimensional manifold endowed with an almost contact structure $(\overline{\phi}, \xi, \eta)$, i.e. $\overline{\phi}$ is a tensor field of type $(1,1)$, ξ is a vector field, and η is a 1-form satisfying

$$\overline{\phi}^2 = -\mathbf{I} + \eta \otimes \xi, \ \eta(\xi) = 1, \ \eta \circ \overline{\phi} = 0, \ \overline{\phi}\xi = 0 \text{ and } \operatorname{rank} \overline{\phi} = 2n. \tag{2.1}$$

Then $(\overline{\phi}, \xi, \eta, \overline{g})$ is called an almost contact metric structure on $\overline{M}$ if $(\overline{\phi}, \xi, \eta)$ is an almost contact structure on $\overline{M}$ and $\overline{g}$ is a semi-Riemannian metric on $\overline{M}$ such that, for any vector field $\overline{X}$, $\overline{Y}$ on $\overline{M}$, it results

$$\overline{g}(\xi, \xi) = \varepsilon = \pm 1, \ \ \eta(\overline{X}) = \varepsilon \overline{g}(\xi, \overline{X}), \ \ \overline{g}(\overline{\phi}\,\overline{X}, \overline{\phi}\,\overline{Y}) = \overline{g}(\overline{X}, \overline{Y}) - \varepsilon\, \eta(\overline{X})\, \eta(\overline{Y}). \tag{2.2}$$

If, moreover, $d\eta(\overline{X}, \overline{Y}) = -\overline{g}(\overline{\phi}\,\overline{X}, \overline{Y})$ and $(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Y} = \overline{g}(\overline{X}, \overline{Y})\xi - \varepsilon\, \eta(\overline{Y})\overline{X}$, where $\overline{\nabla}$ is the Levi-Civita connection for the semi-Riemannian metric $\overline{g}$, we call $\overline{M}$ an indefinite Sasakian manifold. From the first equation of (2.2), ξ is never a lightlike vector field on $\overline{M}$.

Sasakian manifolds with indefinite metrics have been first considered by Takahashi [15]. Their importance for physics have been point out by Duggal [8]. We have two classes of indefinite Sasakian manifolds [8]: ξ is spacelike ($\varepsilon = 1$ and the index of $\overline{g}$ is an even number $\nu = 2r$) and ξ is timelike ($\varepsilon = -1$ and the index of $\overline{g}$ is an odd number $\nu = 2r + 1$).

Takahashi [15] shows that it suffices to consider those indefinite almost contact manifolds with spacelike ξ. Hence, from now on, we shall restrict ourselves to the case ξ spacelike unit vector (that is, $\overline{g}(\xi, \xi) = 1$).

In this case, the equality

$$(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Y} = \overline{g}(\overline{X}, \overline{Y})\xi - \eta(\overline{Y})\overline{X}$$

implies $\overline{\nabla}_{\overline{X}}\xi = -\overline{\phi}(\overline{X})$, ξ is a Killing vector field and $(\overline{\nabla}_{\overline{X}}\eta)\overline{Y} = \overline{g}(\overline{\phi}\,\overline{X}, \overline{Y})$ (see [3]).

A plane section σ in $T_p\overline{M}$ is called a $\overline{\phi}$-section if it is spanned by $\overline{X}$ and $\overline{\phi}\,\overline{X}$, where $\overline{X}$ is a unit tangent vector field orthogonal to ξ. The sectional curvature of a $\overline{\phi}$-section σ is called a $\overline{\phi}$-sectional curvature. A Sasakian manifold $\overline{M}$ with constant $\overline{\phi}$-sectional curvature c is said to be a *Sasakian space form* and is denoted by $\overline{M}(c)$. The curvature tensor $\overline{R}$ of a Sasakian space form $\overline{M}(c)$ is given in [16]: for any $\overline{X}$, $\overline{Y}$, $\overline{Z} \in \Gamma(T\overline{M})$, we have

$$\begin{aligned} \overline{R}(\overline{X}, \overline{Y})\overline{Z} = {} & \frac{c+3}{4}\left(\overline{g}(\overline{Y}, \overline{Z})\overline{X} - \overline{g}(\overline{X}, \overline{Z})\overline{Y}\right) + \frac{c-1}{4}\big(\eta(\overline{X})\eta(\overline{Z})\overline{Y} \\ & - \eta(\overline{Y})\eta(\overline{Z})\overline{X} + \overline{g}(\overline{X}, \overline{Z})\eta(\overline{Y})\xi - \overline{g}(\overline{Y}, \overline{Z})\eta(\overline{X})\xi + \overline{g}(\overline{\phi}\,\overline{Y}, \overline{Z})\overline{\phi}\,\overline{X} \\ & - \overline{g}(\overline{\phi}\,\overline{X}, \overline{Z})\overline{\phi}\,\overline{Y} - 2\overline{g}(\overline{\phi}\,\overline{X}, \overline{Y})\overline{\phi}\,\overline{Z}\big). \end{aligned} \tag{2.3}$$

Let $(\overline{M}, \overline{g})$ be a $(2n+1)$-dimensional semi-Riemannian manifold with index s, $0 < s < 2n+1$, and let (M, g) be a hypersurface of $\overline{M}$, with $g = \overline{g}_{|M}$. M is a lightlike hypersurface of $\overline{M}$ if g is of constant rank $2n-1$ and the normal bundle $TM^{\perp}$ is a distribution of rank 1 on M (cf. [9]). A complementary bundle of $TM^{\perp}$

in TM is a rank $2n-1$ non-degenerate distribution over M. It is called a *screen distribution* and is often denoted by $S(TM)$. A lightlike hypersurface endowed with a specific screen distribution is denoted by the triple $(M, g, S(TM))$. As $TM^\perp$ lies in the tangent bundle, the following Duggal-Bejancu result [9] has an important role in studying the geometry of a lightlike hypersurface.

Theorem 2.1. *Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\overline{M}, \overline{g})$. Then, there exists a unique vector bundle $N(TM)$ of rank 1 over M such that for any non-zero section E of $TM^\perp$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique section N of $N(TM)$ on $\mathcal{U}$ satisfying*

$$g(N, E) = 1 \qquad \textit{and} \qquad \overline{g}(N, N) = \overline{g}(N, W) = 0\ \forall W \in \Gamma(S(TM)|_{\mathcal{U}}).$$

Throughout the paper, all manifolds are supposed to be paracompact and smooth. We denote Γ(E) the smooth sections of the vector bundle E. Also by $\perp$ and $\oplus$ we denote the orthogonal and nonorthogonal direct sum of two vector bundles. By Theorem 2.1 we may write down the following decomposition

$$\begin{aligned} TM &= S(TM) \perp TM^\perp, & (2.4)\\ T\overline{M} &= TM \oplus N(TM) = S(TM) \perp (TM^\perp \oplus N(TM)). & (2.5) \end{aligned}$$

Let $\overline{\nabla}$ be the Levi-Civita connection on $(\overline{M}, \overline{g})$, then by using the first decomposition of (2.5), we have Gauss and Weingarten formulae in the form

$$\begin{aligned} \overline{\nabla}_X Y &= \nabla_X Y + h(X, Y) \quad \text{and} & (2.6)\\ \overline{\nabla}_X V &= -A_V X + \nabla^\perp_X V, \quad \forall X, Y \in \Gamma(TM),\ V \in \Gamma(N(TM)), & (2.7) \end{aligned}$$

where $\nabla_X Y$, $A_V X \in \Gamma(TM)$ and $h(X, Y)$, $\nabla^\perp_X V \in \Gamma(N(TM))$. ∇ is a symmetric linear connection on M called an induced linear connection, $\nabla^\perp$ is a linear connection on the vector bundle $N(TM)$. Moreover, h is a $\Gamma(N(TM))$-valued symmetric bilinear form and A_V is the shape operator of M concerning V.

Equivalently, consider a normalizing pair $\{E, N\}$ as in Theorem 2.1. Then, for all $X, Y \in \Gamma(TM|_{\mathcal{U}})$, (2.6) and (2.7) take the form

$$\begin{aligned} \overline{\nabla}_X Y &= \nabla_X Y + B(X, Y) N & (2.8)\\ \text{and} \quad \overline{\nabla}_X N &= -A_N X + \tau(X) N. \end{aligned}$$

It is important to mention that the second fundamental form B is independent of the choice of screen distribution, in fact, from (2.8), we obtain

$$B(X, Y) = \overline{g}(\overline{\nabla}_X Y, E) \ \text{ and } \ \tau(X) = \overline{g}(\nabla^\perp_X N, E) \quad \forall X,\ Y \in \Gamma(TM|_{\mathcal{U}}).$$

Let P be the projection morphism of TM on $S(TM)$ with respect to the orthogonal decomposition of TM. We have

$$\begin{aligned} \nabla_X PY &= \nabla^*_X PY + C(X, PY) E & (2.9)\\ \text{and } \nabla_X E &= -A^*_E X - \tau(X) E, & (2.10) \end{aligned}$$

where $\nabla^*_X PY$ and $A^*_E X$ belong to $\Gamma(S(TM))$. C, A^*_E and ∇^* are called the local second fundamental form, the local shape operator and the induced connection on $S(TM)$. The induced linear connection ∇ is not a metric connection and we have

$$(\nabla_X g)(Y,Z) = B(X,Y)\theta(Z) + B(X,Z)\theta(Y), \quad \forall X,\ Y \in \Gamma(TM|_{\mathcal{U}}),$$

where θ is a differential 1-form locally defined on M by $\theta(\cdot) := \overline{g}(N,\cdot)$. Also, we have the following identities,

$$g(A^*_E X, PY) = B(X,PY), \quad g(A^*_E X, N) = 0, \quad B(X,E) = 0.$$

Finally, using (2.8), the curvature tensor fields $\overline{R}$ and R of $\overline{M}$ and M respectively, are related as

$$\begin{aligned}\overline{R}(X,Y)Z &= R(X,Y)Z + B(X,Z)A_N Y - B(Y,Z)A_N X \\ &+ ((\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z))N,\end{aligned} \tag{2.11}$$

where

$$(\nabla_X B)(Y,Z) = X.B(Y,Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z). \tag{2.12}$$

In general, the screen distribution is not necessarily integrable (see [9]). More precisely, the following statement holds.

Theorem 2.2. **(Duggal-Bejancu)** *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\overline{M}, \overline{g})$. Then, the following assertions are equivalent:*

(i) *$S(TM)$ is an integrable distribution;*
(ii) *$C(X,Y) = C(Y,X)$, $\forall X, Y \in \Gamma(S(TM))$;*
(iii) *The shape operator of M is symmetric with respect to g.*

3. Lightlike Hypersurfaces of Indefinite Sasakian Manifolds

Let $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$ be an indefinite Sasakian manifold and (M, g) be its lightlike hypersurface, *tangent to the structure vector field ξ ($\xi \in TM$)*. If E is a local section of $TM^\perp$, then $\overline{g}(\overline{\phi}E, E) = 0$, and $\overline{\phi}E$ is tangent to M. Thus $\overline{\phi}(TM^\perp)$ is a distribution on M of rank 1 such that $\overline{\phi}(TM^\perp) \cap TM^\perp = \{0\}$. This enables us to choose a screen distribution $S(TM)$ such that it contains $\overline{\phi}(TM^\perp)$ as vector subbundle. We consider a local section N of $N(TM)$. Since $\overline{g}(\overline{\phi}N, E) = -\overline{g}(N, \overline{\phi}E) = 0$, we deduce that $\overline{\phi}N$ is also tangent to M and belongs to $S(TM)$. On the other hand, since $\overline{g}(\overline{\phi}N, N) = 0$, we see that the component of $\overline{\phi}N$ with respect to E vanishes. Thus $\overline{\phi}N \in \Gamma(S(TM))$. From (2.1), we have $\overline{g}(\overline{\phi}N, \overline{\phi}E) = 1$. Therefore, $\overline{\phi}(TM^\perp) \oplus \overline{\phi}(N(TM))$ (direct sum but not orthogonal) is a non-degenerate vector subbundle of $S(TM)$ of rank 2. It is known (cf. [5]) that if M is tangent to the structure vector field ξ, then, *ξ belongs to $S(TM)$*. Using this, and since $\overline{g}(\overline{\phi}E, \xi) = \overline{g}(\overline{\phi}N, \xi) = 0$, there exists a non-degenerate distribution D_0 of rank $2n-4$ on M such that

$$S(TM) = \left(\overline{\phi}(TM^\perp) \oplus \overline{\phi}(N(TM))\right) \perp D_0 \perp \langle\xi\rangle, \tag{3.1}$$

where $\langle\xi\rangle$ is the distribution spanned by ξ, that is, $\langle\xi\rangle = Span\{\xi\}$. The distribution D_0 is invariant under $\overline{\phi}$, that is, $\overline{\phi}(D_0) = D_0$.

Moreover, from (2.4) and (3.1) we obtain the decomposition

$$\begin{aligned} TM &= \left(\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))\right) \perp D_0 \perp < \xi > \perp TM^{\perp}, \\ T\overline{M} &= \left(\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))\right) \perp D_0 \perp < \xi > \perp (TM^{\perp} \oplus N(TM)). \end{aligned}$$

Now, we consider the distributions on M

$$D := TM^{\perp} \perp \overline{\phi}(TM^{\perp}) \perp D_0, \qquad D' := \overline{\phi}(N(TM)).$$

Then, D is invariant under $\overline{\phi}$ and

$$TM = D \oplus D' \perp \langle\xi\rangle. \tag{3.2}$$

Let us consider the local lightlike vector fields $U := -\overline{\phi}N$, $V := -\overline{\phi}E$. Then, from (3.2), any X on M is written as $X = RX + QX + \eta(X)\xi$, $QX = u(X)U$, where R and Q are the projection morphisms of TM into D and D', respectively, and u is a differential 1-form locally defined on M by $u(\cdot) := g(V, \cdot)$. Applying $\overline{\phi}$ to X and (2.1) , we obtain

$$\overline{\phi}X = \phi X + u(X)N,$$

where ϕ is a tensor field of type (1,1) defined on M by $\phi X := \overline{\phi}RX$ and we also have

$$\phi^2 X = -X + \eta(X)\xi + u(X)U, \ \forall X \in \Gamma(TM). \tag{3.3}$$

By using (2.1) we derive

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) - u(Y)v(X) - u(X)v(Y),$$

where v is 1-form locally defined on M by $v(\cdot) = g(U, \cdot)$. We note that

$$g(\phi X, Y) + g(X, \phi Y) = -u(X)\theta(Y) - u(Y)\theta(X).$$

For any X, $Y \in \Gamma(TM)$, we have the following useful identities:

$$\nabla_X \xi = -\phi X, \tag{3.4}$$

$$B(X,\xi) = -u(X), \tag{3.5}$$

$$C(X,\xi) = -v(X), \tag{3.6}$$

$$B(X,U) = C(X,V), \tag{3.7}$$

$$(\nabla_X u)Y = -B(X,\phi Y) - u(Y)\tau(X). \tag{3.8}$$

Proposition 3.1. *Let $(M,\, g,\, S(TM))$ be a lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. For any X, $Y \in \Gamma(TM)$, the Lie derivative of g with respect to the vector field V is given by*

$$(L_V g)(X,Y) = X.u(Y) + Y.u(X) + u([X,Y]) - 2u(\nabla_X Y). \tag{3.9}$$

Proof. From a straightforward calculation we complete the proof. □

Let $\overline{M}(c)$ be an indefinite Sasakian space form and M be a lightlike hypersurface of $\overline{M}(c)$. Let us consider the pair $\{E, N\}$ on $\mathcal{U} \subset M$ (see Theorem 2.1) and by using (2.11), we obtain

$$\begin{aligned}(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) &= \tau(Y)B(X,Z) - \tau(X)B(Y,Z)\\ &+ \frac{c-1}{4}\left(\overline{g}(\overline{\phi}Y,Z)u(X) - \overline{g}(\overline{\phi}X,Z)u(Y) - 2\overline{g}(\overline{\phi}X,Y)u(Z)\right).\end{aligned} \tag{3.10}$$

Theorem 3.2. *Let M be a lightlike hypersurface of an indefinite Sasakian space form $\overline{M}(c)$ of constant curvature c with $\xi \in TM$. Then, the Lie derivative of the second fundamental form B with respect to ξ is given by*

$$(L_\xi B)(X,Y) = -\tau(\xi)B(X,Y), \quad \forall\, X, Y \in \Gamma(TM). \tag{3.11}$$

Proof. Using (2.12) and (3.4), we obtain

$$(\nabla_\xi B)(X,Y) = (L_\xi B)(X,Y) + B(\phi X,Y) + B(X,\phi Y). \tag{3.12}$$

Similarly, using (2.12), (3.4) and (3.5), we have

$$(\nabla_X B)(\xi,Y) = -X.u(Y) + B(\phi X,Y) + u(\nabla_X Y). \tag{3.13}$$

Subtracting (3.12) and (3.13), and using (3.8) we obtain

$$(\nabla_\xi B)(X,Y) - (\nabla_X B)(\xi,Y) = (L_\xi B)(X,Y) - u(Y)\tau(X). \tag{3.14}$$

From (3.10) and after calculation, the left hand side of (3.14) becomes

$$(\nabla_\xi B)(X,Y) - (\nabla_X B)(\xi,Y) = -u(Y)\tau(X) - \tau(\xi)B(X,Y). \tag{3.15}$$

The expressions (3.14) and (3.15) imply $(L_\xi B)(X,Y) = -\tau(\xi)B(X,Y)$. □

From (2.3) and (2.11), a direct calculation shows that

$$\begin{aligned}&(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) + \tau(Y)C(X,PZ) - \tau(X)C(Y,PZ)\\ &= \frac{c+3}{4}\left(\overline{g}(Y,PZ)\theta(X) - \overline{g}(X,PZ)\theta(Y)\right) + \frac{c-1}{4}\left(\eta(X)\eta(PZ)\theta(Y)\right.\\ &\quad -\eta(Y)\eta(PZ)\theta(X) + \overline{g}(\overline{\phi}Y,PZ)v(X) - \overline{g}(\overline{\phi}X,PZ)v(Y)\\ &\quad \left.-2\overline{g}(\overline{\phi}X,Y)v(PZ)\right).\end{aligned} \tag{3.16}$$

Lemma 3.3. *Let $(M,\, g,\, S(TM))$ be a lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M},\overline{g})$ with $\xi \in TM$. For any $X,\, Y \in \Gamma(TM)$ it results*

$$g(\nabla_X U,Y) + u(A_N X)\theta(Y) = -C(X,\phi Y) - \theta(X)\eta(Y) + \tau(X)v(Y). \tag{3.17}$$

Proof. By straightforward calculation and also by using (2.8) and (2.9) we obtain

$$g(\nabla_X U,Y) + u(A_N X)\theta(Y) = -\overline{g}(A_N X,\overline{\phi}Y) - \theta(X)\eta(Y) + \tau(X)v(Y)$$

which completes the proof. □

Lemma 3.4. *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. Then, the covariant derivative of v and the Lie derivative of g with respect to the vector field U are given, respectively, by*

$$(\nabla_X v)Y = -C(X, \phi Y) - \theta(X)\eta(Y) + \tau(X)v(Y), \tag{3.18}$$

$$(L_U g)(X, Y) = X.v(Y) + Y.v(X) + v([X, Y]) - 2v(\nabla_X Y), \tag{3.19}$$

for any X, $Y \in \Gamma(TM)$.

Proof. The proof of (3.18) follows from (3.17), while (3.19) follows from direct calculations. □

4. Screen Integrable Lightlike Hypersurfaces of Indefinite Sasakian Manifolds

Let M be a lightlike hypersurface of an indefinite Sasakian space form $\overline{M}(c)$ with $\xi \in TM$. From the differential geometry of lightlike hypersurfaces, we recall the following desirable property for lightlike geometry. It is known that lightlike submanifolds whose screen distribution is integrable have interesting properties. Now, we study integrable distributions with specific attention to the screen distribution $S(TM)$ and the distribution $D_0 \perp \langle\xi\rangle$.

By Theorem 2.2, the screen distribution $S(TM)$ of M is integrable if and only if the second fundamental form of $S(TM)$ is symmetric on $\Gamma(S(TM))$. However, we have

$$u([X, Y]) = B(X, \phi Y) - B(\phi X, Y) \quad \text{for any } X, Y \in \Gamma(D \perp \langle\xi\rangle).$$

So, it is very easy to see that the distribution $D \perp \langle\xi\rangle$ is integrable if and only if $B(X, \phi Y) = B(\phi X, Y)$.

Proposition 4.1. *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Sasakian space form $\overline{M}(c)$ with $\xi \in TM$. If the screen distribution $S(TM)$ is integrable, then it results*

$$(L_\xi C)(X, PY) = \tau(\xi)C(X, PY) \quad \textit{for any } X, Y \in \Gamma(TM). \tag{4.1}$$

Proof. If the screen distribution $S(TM)$ of a lightlike hypersurface M is integrable, then, from (3.16) and using (3.6), for any X, $Y \in \Gamma(TM)$, we have

$$\begin{aligned}(\nabla_\xi C)(X, PY) - (\nabla_X C)(\xi, PY) = &- \eta(PY)\theta(X) + \tau(X)v(PY)\\ &+ \tau(\xi)C(X, PY).\end{aligned} \tag{4.2}$$

On the other hand, using (3.6) and (3.17), we have

$$\begin{aligned}(\nabla_\xi C)(X, PY) &= \xi.C(X, PY) - C(\nabla_\xi X, PY) - C(X, \nabla_\xi(PY))\\ &= (L_\xi C)(X, PY) + C(\phi X, PY) + C(X, \phi PY),\end{aligned} \tag{4.3}$$

$$\begin{aligned}(\nabla_X C)(\xi, PY) &= -X.v(PY) + C(\phi X, PY) + v(\nabla_X PY)\\ &= C(X, \phi PY) + \theta(X)\eta(PY) - \tau(X)v(PY) + C(\phi X, PY).\end{aligned} \tag{4.4}$$

Putting (4.3) and (4.4) together in (4.2), we obtain (4.1). □

Let us assume that the screen distribution $S(TM)$ of M is integrable and let M' be a leaf of $S(TM)$. Then, using (2.8) and (2.9), we obtain

$$\begin{aligned} \overline{\nabla}_X Y &= \nabla^*_X Y + C(X,Y)E + B(X,Y)N \\ &= \nabla'_X Y + h'(X,Y), \quad \forall X,\, Y \in \Gamma(TM'), \end{aligned} \tag{4.5}$$

where ∇' and h' are the Levi-Civita connection and second fundamental form of M' in $\overline{M}$. Thus,

$$h'(X,Y) = C(X,Y)E + B(X,Y)N, \quad \forall X,\, Y \in \Gamma(TM'). \tag{4.6}$$

In the sequel, we need the following lemma

Lemma 4.2. *Let $(M, g, S(TM))$ be a screen integrable lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$ and M' be a leaf of $S(TM)$. Then, for any $X \in \Gamma(TM')$,*

$$\nabla'_X \xi = -\phi X + v(X)E, \tag{4.7}$$

$$\nabla'_X U = -v(A_N X)E - v(A^*_E X)N + \overline{\phi}(A_N X) + \tau(X)U, \tag{4.8}$$

$$\nabla'_X V = -u(A_N X)E - u(A^*_E X)N + \overline{\phi}(A^*_E X) - \tau(X)V. \tag{4.9}$$

Proof. From a straightforward calculation we complete the proof. □

It is well known that the second fundamental form and the shape operators of a non-degenerate hypersurface (in general, submanifold) are related by means of the metric tensor field. Contrary to this, we see from (2.9) and (2.10), in the case of lightlike hypersurfaces, the second fundamental forms on M and their screen distribution $S(TM)$ are related to their respective shape operators A_N and A^*_E. As the shape operator is an information tool in studing the geometry of submanifolds, their studying turns out very important. For instance, in [10] a class of lightlike hypersurfaces whose shape operators are the same as the one of their screen distribution up to a conformal non zero smooth factor in $\mathcal{F}(M)$ was considered. That work gave a way to generate, under some geometric conditions, an integrable canonical screen (see [10] for more details).

Next, we study these operators and give their implications in lightlike hypersurface of indefinite Sasakian manifolds with $\xi \in TM$.

Proposition 4.3. *Let $(M, g, S(TM))$ be a screen integrable lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$ and M' be a leaf of $S(TM)$. Then, we have*

(i) *The vector field U is parallel with respect to the Levi-Civita connection ∇' on M' if and only if*

$$A_N X = \eta(A_N X)\xi + v(A_N X)V + u(A_N X)U \quad \forall X \in \Gamma(TM')$$

and τ vanishes on M';

(ii) *The vector field V is parallel with respect to the Levi-Civita connection ∇' on M' if and only if*

$$A_E^* X = \eta(A_E^* X)\xi + v(A_E^* X)V + u(A_E^* X)U \quad \forall X \in \Gamma(TM')$$

and τ vanishes on M'.

Proof. (i) Suppose U is parallel with respect to the Levi-Civita connection ∇' on M'. Then, by using (4.8), we have

$$\overline{\phi}(A_N X) = v(A_N X)E + v(A_E^* X)N - \tau(X)U \qquad \text{for any } X \in \Gamma(TM').$$

Since $\overline{\phi}(A_N X) = \phi(A_N X) + u(A_N X)N$, by using (3.7), we obtain

$$\phi(A_N X) = v(A_N X)E - \tau(X)U. \tag{4.10}$$

Apply ϕ to (4.10) and by using (3.3) and the fact that $\phi U = 0$, we obtain

$$A_N X = \eta(A_N X)\xi + u(A_N X)U + v(A_N X)V. \tag{4.11}$$

Putting (4.11) in (4.8) and using (3.7), we get $\tau(X) = 0$. The converse is obvious. In the similar way, by using (4.9) the assertion (ii) follows. □

Corollary 4.4. *Let $(M, g, S(TM))$ be a screen integrable lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$ and let M' be a leaf of $S(TM)$ such U and V are parallel with respect to the Levi-Civita connection ∇' on M'. Then, the type number $t'(x)$ of M' (with $x \in M'$) satisfies $t'(x) \leq 3$.*

Proof. The proof follows from Proposition 4.3. □

Let W be an element of $\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))$ which is a non-degenerate vector subbundle of $S(TM)$ of rank 2. Then, there exist non-zero functions a and b such that

$$W = aV + bU.$$

It is easy to check that $a = v(W)$ and $b = u(W)$. Let ω be a 1-form locally defined by $\omega(\cdot) = g(W, \cdot)$.

Lemma 4.5. *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. Then, the covariant derivative of ω and the Lie derivative of g with respect to the vector field W are given, respectively, by*

$$(\nabla_X \omega)Y = -v(W)B(X, \phi Y) - u(W)\left(C(X, \phi Y) + \theta(X)\eta(Y)\right), \tag{4.12}$$

$$(L_W g)(X, Y) = X.\omega(Y) + Y.\omega(X) + \omega([X, Y]) - 2\omega(\nabla_X Y), \tag{4.13}$$

for any X, $Y \in \Gamma(TM)$.

Proof. Using (3.8) and (3.18), for any X, $Y \in \Gamma(TM)$ we obtain

$$\begin{aligned}(\nabla_X \omega)Y &= u(Y)(\nabla_X v)W + v(W)(\nabla_X u)Y + v(Y)(\nabla_X u)W + u(W)(\nabla_X v)Y \\ &= -v(W)B(X, \phi Y) - u(W)(C(X, \phi Y) + \theta(X)\eta(Y))\end{aligned}$$

which proves (4.12), while (4.13) follows from a direct calculation. □

Lemma 4.6. *Let $(M, g, S(TM))$ be a screen integrable lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$ and let M' be a leaf of $S(TM)$. Then, for any X, $Y \in \Gamma(TM')$, it results*

$$\omega(\nabla'_X Y) = -\omega(\overline{\phi}h'(X, \phi Y)), \tag{4.14}$$

$$\omega([X,Y]) = \omega(\overline{\phi}h'(\phi X, Y) - \overline{\phi}h'(X, \phi Y)). \tag{4.15}$$

Proof. Using (4.5) and(4.6), for any X, $Y \in \Gamma(TM')$ we obtain

$$\begin{aligned} \omega(\nabla'_X Y) &= g(W, \nabla'_X Y) = \overline{g}(W, \overline{\nabla}_X Y) = v(W)u(\overline{\nabla}_X Y) + u(W)v(\overline{\nabla}_X Y) \\ &= v(W)B(X, \phi Y) + u(W)C(X, \phi Y) = -\omega(\overline{\phi}h'(X, \phi Y)) \end{aligned}$$

and

$$\omega([X,Y]) = \omega(\nabla'_X Y) - \omega(\nabla'_Y X) = -\omega(\overline{\phi}h'(X, \phi Y) - \overline{\phi}h'(Y, \phi X)),$$

which complete the proof. □

We report the following result proved in [11].

Theorem 4.7. *Let $(M,\, g,\, S(TM))$ be a lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. Then, the distribution $D_0 \perp \langle\xi\rangle$ is integrable if and only if for all X, $Y \in \Gamma(D_0 \perp \langle\xi\rangle)$ it results*

$$\begin{aligned} C(\phi X, Y) &= C(X, \phi Y), \\ B(\phi X, Y) &= B(X, \phi Y), \\ C(X,Y) &= C(Y,X). \end{aligned}$$

Theorem 4.8. *Let $(M,\, g,\, S(TM))$ be a lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. Suppose the distribution $D_0 \perp \langle\xi\rangle$ is integrable. Let M' be a leaf of $D_0 \perp \langle\xi\rangle$. Then,*

(i) *If M' is totally geodesic in M, then M' is auto-parallel with respect to the Levi-Civita connection ∇' in M and $\overline{\phi}(TM^\perp) \oplus \overline{\phi}(N(TM))$ is a Killing distribution on M';*

(ii) *If M' is parallel with respect to the Levi-Civita connection ∇' in M, then $\overline{\phi}(TM^\perp)$ and $\overline{\phi}(N(TM))$ are Killing distribution on M'.*

Proof. (i) Writing $Y \in \Gamma(D_0 \perp \langle\xi\rangle)$ as

$$Y = \sum_{i=1}^{2n-4} \frac{g(Y, F_i)}{g(F_i, F_i)} F_i + \eta(Y)\xi,$$

where $g(F_i, F_i) \neq 0$ and $\{F_i\}_{1 \leq i \leq 2n-4}$ is an orthogonal basis of D_0. So, it is easy to check that, for any X, $Y \in \Gamma(TM')$, we have

$$h'(X, \phi Y) = \sum_{i=1}^{2n-4} \frac{g(Y, F_i)}{g(F_i, F_i)} h'(X, \phi F_i).$$

If M' is totally geodesic, then, for any X, $Y \in \Gamma(D_0 \perp \langle\xi\rangle)$, it is $h'(X,Y) = 0$. In particular,

$$h'(X, \phi Y) = \sum_i \frac{g(Y, F_i)}{g(F_i, F_i)} h'(X, \phi F_i) = 0.$$

The auto-parallelism of M' follows from (4.14). Using (4.13), (4.14), (4.15) and the fact that $\omega(X) = 0$, $\forall X \in \Gamma(D_0 \perp \langle\xi\rangle)$, we obtain $(L_W g)(X,Y) = 0$. So $\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))$ is a Killing distribution on M'.
(ii) If M' is parallel with respect to the connection in M, then, for any X, $Y \in \Gamma(TM')$, $(\nabla'_X h')Y = 0$. That is, $(\nabla'_X C)Y - C(X,Y)\tau(X) = 0$ and $(\nabla'_X B)Y + B(X,Y)\tau(X) = 0$. Using (2.12), (3.4), (3.5), (3.8) and (3.11), since $\overline{\phi}(TM^{\perp}) \perp (D_0 \perp \langle\xi\rangle)$ and $D_0 \perp \langle\xi\rangle$ integrable, we have

$$\begin{aligned} 0 &= (\nabla'_\xi B)(X,Y) + \tau(\xi)B(X,Y) \\ &= B(\phi X, Y) + B(X, \phi Y) = -(L_V g)(X,Y). \end{aligned}$$

Also, using (4.1) and since $\overline{\phi}(N(TM)) \perp (D_0 \perp \langle\xi\rangle)$, for any X, $Y \in \Gamma(TM')$, we obtain

$$\begin{aligned} 0 &= (\nabla'_\xi C)(X,Y) - \tau(\xi)C(X,Y) \\ &= C(\phi X, Y) + C(X, \phi Y) = -(L_U g)(X,Y), \end{aligned}$$

which completes the proof. □

Note that, the Lie derivative (4.13) can be expressed in functions of Lie derivatives (3.9) and (3.19) as, for any X, $Y \in \Gamma(TM)$, it results

$$\begin{aligned} (L_W g)(X,Y) &= X.v(W)u(Y) + Y.v(W)u(X) + X.u(W)v(Y) + Y.u(W)v(X) \\ &\quad + v(W)(L_V g)(X,Y) + u(W)(L_U g)(X,Y). \end{aligned}$$

Theorem 4.9. *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. Suppose the distribution $D_0 \perp \langle\xi\rangle$ is integrable. Let M' be a leaf of $D_0 \perp \langle\xi\rangle$. Then, $\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))$ is a Killing distribution on M' if and only if $\overline{\phi}(TM^{\perp})$ and $\overline{\phi}(N(TM))$ are Killing distributions on M'.*

5. Totally Contact Umbilical Leaf of Integrable Screen Distributions

In this section, we deal with the geometry of the mean curvature vector of a leaf of an integrable screen distribution of a lightlike hypersurface M of an indefinite Sasakian space form $\overline{M}(c)$ by introducing a new concept. First of all, a submanifold M is said to be totally umbilical lightlike hypersurface of a semi-Riemannian manifold $\overline{M}$ if the local second fundamental form B of M satisfies

$$B(X,Y) = \rho g(X,Y), \quad \forall X,\ Y \in \Gamma(TM),$$

where ρ is a smooth function on $\mathcal{U} \subset M$.

If we assume that M is totally umbilical lightlike hypersurface of an indefinite Sasakian manifold $\overline{M}$, the Gauss formula (2.6) implies that

$$\overline{\phi}X = -\overline{\nabla}_X\xi = -\nabla_X\xi - B(X,\xi)N$$

and since $\overline{\phi}(\xi) = 0$, we have $B(\xi,\xi) = 0$. Being also $B(X,Y) = \rho g(X,Y)$, for any X, $Y \in \Gamma(TM)$, we get $0 = B(\xi,\xi) = \rho$. Hence, M is totally geodesic. Also, $\overline{\phi}X = \phi X - \rho\eta(X)N = \phi X$, that is M is invariant in $\overline{M}$. Therefore, we can state the following result.

Proposition 5.1. *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. If M is totally umbilical, then M is totally geodesic and invariant.*

It follows from the Proposition 5.1 that a Sasakian $\overline{M}(c)$ does not admit any non-totally geodesic, totally umbilical lightlike hypersurface. From this point of view, Bejancu [2] considered the concept of totally contact umbilical semi-invariant submanifolds. The notion of totally contact umbilical submanifolds was first defined by Kon [12]. As the notion of totally contact geodesic submanifolds of Sasakian manifolds corresponds to that of totally geodesic submanifolds of Kaehlerian manifolds, it is important to investigate the parallelism of the nonzero mean curvature vector. The terminology of *extrinsic sphere* [7] also is going to be used in case of totally contact geodesic submanifolds.

We say that a totally contact umbilical submanifold is an *extrinsic sphere* when it has parallel non zero mean curvature vector (see [7]). In [13], the author showed that if M is a totally contact umbilical lightlike hypersurface of $\overline{M}(c)$ with $\xi \in TM$, that is, the second fundamental form h of M satisfies

$$h(X,Y) = H\,(g(X,Y) - \eta(X)\eta(Y)) + \eta(X)B(Y,\xi) + \eta(Y)B(X,\xi), \qquad (5.1)$$

where $H = \lambda N$ normal vector field and λ is a smooth function on $\mathcal{U} \subset M$, then λ satisfies the partial differential equations

$$E \cdot \lambda + \lambda\tau(E) - \lambda^2 = 0 \qquad (5.2)$$

$$\text{and} \qquad PX \cdot \lambda + \lambda\tau(PX) = 0, \quad \forall\, X \in \Gamma(TM). \qquad (5.3)$$

These equations are similar to those of the indefinite Kaehlerian case (see [9] for more details). However, there are nontrivial differences arising in the details of the proof in [13]. We also note that the partial differential equations (5.2) and the modified (5.3), $PX \cdot \lambda + \lambda\tau(PX) = 0$ with $PX \in \Gamma(S(TM) - \langle\xi\rangle)$ (that is, we exclude the partial differential equation in terms of ξ) arise when the submanifold M is a $D\oplus D'$-totally umbilical lightlike hypersurface, that is, $B(X,Y) = \rho g(X,Y)$, for any X, $Y \in \Gamma(D \oplus D')$. Because, in the direction of $D \oplus D'$, the function ρ is nowhere vanishing, in general, such a concept is called proper totally umbilical (see [9]).

Note that, if $\lambda = 0$ in (5.1), then the lightlike hypersurface M is said to be totally contact geodesic. The notion of totally contact geodesic submanifolds of

Sasakian manifolds corresponds to that of totally geodesic submanifolds of Kaehlerian manifolds.

Also, from (5.2) and (5.3), we have

$$\nabla^{\perp}_{E}H = \overline{g}(H,E)^2N \qquad \text{and} \qquad \nabla^{\perp}_{PX}H = 0, \quad \forall X \in \Gamma(TM).$$

So, for any $X \in \Gamma(TM)$, we obtain

$$\nabla^{\perp}_{X}H = \nabla^{\perp}_{PX}H + \theta(X)\nabla^{\perp}_{E}H = \theta(X)\overline{g}(H,E)^2N,$$

that is, $\nabla^{\perp}_{X}H \neq 0$, and consequently, we can state the following result.

Theorem 5.2. *Let $\overline{M}(c)$ be an indefinite Sasakian space form and M be a totally contact umbilical (or $D \oplus D'$-totally umbilical) lightlike hypersurface of $\overline{M}(c)$ with $\xi \in TM$. Then, M cannot be an extrinsic sphere.*

Now, we pay attention to a specific example of a screen integrable lightlike hypersurface. We say that the screen distribution $S(TM)$ is totally contact umbilical if the local second fundament form C of $S(TM)$ satisfies

$$C(X,Y) = \alpha\,(g(X,Y) - \eta(X)\eta(Y)) + \eta(X)C(Y,\xi) + \eta(Y)C(X,\xi), \tag{5.4}$$

where α is a smooth function on $\mathcal{U} \subset M$. If we assume that the screen distribution of the lightlike hypersurface M of an indefinite Sasakian manifold, with $\xi \in TM$, is totally contact umbilical, then it follows that C is symmetric on $\Gamma(S(TM))$ and hence, by Theorem 2.2, the distribution $S(TM)$ is integrable.

Theorem 5.3. *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Sasakian space form $\overline{M}(c)$ with $\xi \in TM$ such that $S(TM)$ is totally contact umbilical. Then, $S(TM)$ is totally contact geodesic and $c = -3$.*

Proof. By a direct calculation of the right hand side in (3.16) and using (5.4), we get

$$\begin{aligned}
&(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) + \tau(Y)C(X,PZ) - \tau(X)C(Y,PZ)\\
&\quad = (g(Y,PZ) - \eta(Y)\eta(PZ))\,X.\alpha - (g(X,Z) - \eta(X)\eta(PZ))\,Y.\alpha\\
&\qquad + \alpha\,(B(X,PZ)\theta(Y) - B(Y,PZ)\theta(X))\\
&\qquad + \alpha\,(u(X)\theta(Y) + g(\phi X,Y) - u(Y)\theta(X) - g(\phi Y,X))\,\eta(PZ)\\
&\qquad + \alpha\,(g(\phi X,PZ)\eta(Y) - g(\phi Y,PZ)\eta(X))\\
&\qquad + (u(X)\theta(Y) + g(\phi X,Y) - u(Y)\theta(X) - g(\phi Y,X))\,v(PZ)\\
&\qquad + (g(\nabla_Y U,PZ)\eta(X) - g(\nabla_X U,PZ)\eta(Y))\\
&\qquad + (g(\phi X,PZ)v(Y) - g(\phi Y,PZ)v(X))\\
&\qquad + (B(Y,U)\theta(X) + g(\nabla_Y U,X) - B(X,U)\theta(Y)\\
&\qquad - g(\nabla_X U,Y))\,\eta(PZ) + \tau(Y)C(X,PZ) - \tau(X)C(Y,PZ).
\end{aligned} \tag{5.5}$$

Putting $X = E$ in (5.5) and in the right hand side of (3.16), we obtain

$$\begin{aligned}(g(Y,PZ) - \eta(Y)\eta(PZ))(E.\alpha) - \alpha B(Y,PZ) - 2\alpha u(Y)\eta(PZ) \\ - \alpha u(PZ)\eta(Y) - 2u(Y)v(PZ) - g(\nabla_E U, PZ)\eta(Y) - u(PZ)v(Y) \\ + (B(Y,U) + g(\nabla_Y U, E) - g(\nabla_E U, Y))\,\eta(PZ) - \tau(E)C(Y,PZ) \\ = \frac{c+3}{4}\overline{g}(Y,PZ) + \frac{c-1}{4}(-\eta(Y)\eta(PZ) + u(PZ)v(Y) + 2u(Y)v(PZ)).\end{aligned} \tag{5.6}$$

Replacing $Y = PZ = U$ in (5.6), we have

$$\overline{g}(\overline{R}(E,U)U,N) = -\alpha B(U,U) = -\alpha C(U,V) = -\alpha^2 = 0.$$

The last assertion is obtained by taking $Y = V$ and $PZ = U$ in (5.6). □

Corollary 5.4. *There exist no lightlike hypersurfaces M of indefinite Sasakian space forms $\overline{M}(c)$ $(c \neq -3)$ with $\xi \in TM$ and totally contact umbilical screen distribution.*

Theorem 5.5. *Let $(M, g, S(TM))$ be a screen integrable lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. Suppose any leaf M' of $S(TM)$ is totally contact umbilical immersed in $\overline{M}$ as non-degenerate submanifold. Then, M' is an extrinsic sphere.*

Proof. By combining the first equations of (2.8) and (2.9), we obtain

$$\begin{aligned}\overline{\nabla}_X Y &= \nabla^*_X Y + C(X,Y)E + B(X,Y)N \\ &= \nabla'_X Y + h'(X,Y), \quad \forall X,\, Y \in \Gamma(TM').\end{aligned} \tag{5.7}$$

Denote by H' the mean curvature vector of M'. As $N(TM) \oplus TM^{\perp}$ is the normal bundle of M', there exist smooth functions λ and ρ such that $H' = \lambda E + \rho N$. Since M' is totally contact umbilical immersed in $\overline{M}$, we have

$$h'(X,Y) = (g(X,Y) - \eta(X)\eta(Y))\,H' + \eta(X)h'(Y,\xi) + \eta(Y)h'(X,\xi).$$

So, from (5.7) we obtain

$$\begin{aligned}\overline{\nabla}_X Y &= \nabla'_X Y + (g(X,Y) - \eta(X)\eta(Y))\,H' \\ &\quad - (\eta(X)v(Y) + \eta(Y)v(X))\,E - (\eta(X)u(Y) + \eta(Y)u(X))\,N\end{aligned}$$

which implies

$$\begin{aligned}\overline{\nabla}_X \overline{\nabla}_Y Z = {} & \nabla'_X \nabla'_Y Z + (g(X, \nabla'_Y Z) - \eta(X)\eta(\nabla'_Y Z))\,H' \\ & + (g(\nabla'_X Y, Z) + g(Y, \nabla'_X Z) - g(\nabla'_X \xi, Y)\eta(Z) - g(\xi, \nabla'_X Y)\eta(Z) \\ & - \eta(Y)g(\nabla'_X \xi, Z) - \eta(Y)g(\xi, \nabla'_X Z))\,H' \\ & + (g(Y,Z) - \eta(Y)\eta(Z))\,\overline{\nabla}_X H' - (\eta(X)v(\nabla'_Y Z) + \eta(\nabla'_Y Z)v(X) \\ & + X.\eta(Y)v(Z) + \eta(Y)X.v(Z) + X.\eta(Z)v(Y) + \eta(Z)X.v(Y))\,E \\ & - (\eta(Y)v(Z) + \eta(Z)v(Y))\,\overline{\nabla}_X E - (\eta(X)u(\nabla'_Y Z) + \eta(\nabla'_Y Z)u(X) \\ & + X.\eta(Y)u(Z) + \eta(Y)X.u(Z) + X.\eta(Z)u(Y) + \eta(Z)X.u(Y))\,N \\ & - (\eta(Y)u(Z) + \eta(Z)u(Y))\,\overline{\nabla}_X N.\end{aligned} \tag{5.8}$$

Also, we have

$$\overline{\nabla}_{[X,Y]}Z = \nabla'_{[X,Y]}Z + (g([X,Y],Z) - \eta([X,Y])\eta(Z))\,H' - (\eta([X,Y])v(Z) + \eta(Z)v([X,Y]))\,E - (\eta([X,Y])u(Z) + \eta(Z)u([X,Y]))N. \tag{5.9}$$

From (5.8), (5.9) and (4.7)-(4.9), after calculations, we obtain

$$\begin{aligned}\overline{R}(X,Y)Z &= R'(X,Y)Z + (g(\phi X,Y)\eta(Z) + g(\phi X,Z)\eta(Y) - g(\phi Y,X)\eta(Z)\\ &\quad - g(\phi Y,Z)\eta(X))\,H' + (g(Y,Z) - \eta(Y)\eta(Z))\,\overline{\nabla}_X H'\\ &\quad - (g(X,Z) - \eta(X)\eta(Z))\,\overline{\nabla}_Y H' - (-\eta(X)\tau(Y)v(Z) + v(X)g(\phi Y,Z)\\ &\quad + v(Z)[g(\phi Y,X) - g(\phi X,Y)] + \eta(Y)\tau(X)v(Z) - v(Y)g(\phi X,Z)\\ &\quad + \eta(Z)[\tau(X)v(Y) - \tau(Y)v(X)])\,E - (\eta(Y)v(Z) + \eta(Z)v(Y))\,\overline{\nabla}_X E\\ &\quad + (\eta(X)v(Z) + \eta(Z)v(X))\,\overline{\nabla}_Y E - (\eta(X)\tau(Y)u(Z) + u(X)g(\phi Y,Z)\\ &\quad + u(Z)[g(\phi Y,X) - g(\phi X,Y)] - \eta(Y)\tau(X)u(Z) - u(Y)g(\phi X,Z)\\ &\quad + \eta(Z)[\tau(Y)u(X) - \tau(X)u(Y)])\,N - (\eta(Y)u(Z) + \eta(Z)u(Y))\,\overline{\nabla}_X N\\ &\quad + (\eta(X)u(Z) + \eta(Z)u(X))\,\overline{\nabla}_Y N.\end{aligned}$$

Consequently,

$$\begin{aligned}\overline{g}(\overline{R}(X,Y)Z,E) &= (g(\phi X,Y)\eta(Z) + g(\phi X,Z)\eta(Y) - g(\phi Y,X)\eta(Z)\\ &\quad - g(\phi Y,Z)\eta(X))\,\overline{g}(H',E) + (g(Y,Z) - \eta(Y)\eta(Z))\,\overline{g}(\overline{\nabla}_X H',E)\\ &\quad - (g(X,Z) - \eta(X)\eta(Z))\,\overline{g}(\overline{\nabla}_Y H',E) + u(Z)\,(g(\phi X,Y) - g(X,\phi Y))\\ &\quad + u(Y)g(\phi X,Z) - u(X)g(\phi Y,Z),\end{aligned} \tag{5.10}$$

$$\begin{aligned}\overline{g}(\overline{R}(X,Y)Z,N) &= (g(\phi X,Y)\eta(Z) + g(\phi X,Z)\eta(Y) - g(\phi Y,X)\eta(Z)\\ &\quad - g(\phi Y,Z)\eta(X))\,\overline{g}(H',N) + (g(Y,Z) - \eta(Y)\eta(Z))\,\overline{g}(\overline{\nabla}_X H',N)\\ &\quad - (g(X,Z) - \eta(X)\eta(Z))\,\overline{g}(\overline{\nabla}_Y H',N) + v(Z)\,(g(\phi X,Y) - g(X,\phi Y))\\ &\quad + v(Y)g(\phi X,Z) - v(X)g(\phi Y,Z).\end{aligned} \tag{5.11}$$

From (5.10) and using (2.3), we obtain

$$\begin{aligned}&\frac{c-1}{4}\left(\overline{g}(\overline{\phi}Y,Z)u(X) - \overline{g}(\overline{\phi}X,Z)u(Y) - 2\overline{g}(\overline{\phi}X,Y)u(Z)\right)\\ &\quad = (g(\phi X,Y)\eta(Z) + g(\phi X,Z)\eta(Y) - g(\phi Y,X)\eta(Z)\\ &\qquad - g(\phi Y,Z)\eta(X))\,\overline{g}(H',E) + (g(Y,Z) - \eta(Y)\eta(Z))\,\overline{g}(\overline{\nabla}_X H',E)\\ &\qquad - (g(X,Z) - \eta(X)\eta(Z))\,\overline{g}(\overline{\nabla}_Y H',E) + u(Z)\,(g(\phi X,Y) - g(X,\phi Y))\\ &\qquad + u(Y)g(\phi X,Z) - u(X)g(\phi Y,Z).\end{aligned} \tag{5.12}$$

Taking $X = \xi$ in this equation, for $Y = U$ and $Z = V$, we have $\overline{g}(\overline{\nabla}_\xi H',E) = 0$. Now, if $X,\ Y,\ Z \in \Gamma(TM' - \xi)$, from (5.12), we have

$$\overline{g}(\overline{\nabla}_X H',E)Y = \overline{g}(\overline{\nabla}_Y H',E)X. \tag{5.13}$$

Similarly, from (5.11) and (2.3), we have

$$\overline{g}(\overline{\nabla}_\xi H', N) = 0 \qquad \text{and} \qquad \overline{g}(\overline{\nabla}_X H', N)Y = \overline{g}(\overline{\nabla}_Y H', N)X. \tag{5.14}$$

Now, suppose that there exists a vector field X_0 on some neighborhood of M' such that $\overline{g}(\overline{\nabla}_{X_0}H', E) \neq 0$ and $\overline{g}(\overline{\nabla}_{X_0}H', N) \neq 0$ at some point p in the neighborhood. From (5.13) and (5.14) it follows that all vectors of the fibre $TM' - \xi$ are collinear with $X_0|_p$. This contradicts $\dim(TM' - \xi) > 1$. This implies $\overline{g}(\overline{\nabla}_X H', E) = 0$ and $\overline{g}(\overline{\nabla}_X H', N) = 0$, $\forall X \in \Gamma(TM' - \xi)$. Since $\overline{g}(\overline{\nabla}_\xi H', E) = 0$ and $\overline{g}(\overline{\nabla}_\xi H', N) = 0$, so, we have $\overline{g}(\overline{\nabla}_X H', E) = 0$ and $\overline{g}(\overline{\nabla}_X H', N) = 0$, $\forall X \in \Gamma(TM')$. These lead, respectively, to $g(\nabla'^{\perp}_X H', E) = 0$ and $g(\nabla'^{\perp}_X H', N) = 0$, where $\nabla'^{\perp}$ is a linear connection on $N(TM) \oplus TM^{\perp}$ defined by $\nabla'^{\perp}_X E = \nabla^{*\perp}_X E = -\tau(X)E$ and $\nabla'^{\perp}_X N = \nabla^{\perp}_X N = \tau(X)N$, which completes the proof. □

Note that, if at each point $p \in M$ we choose a connected open set G on M such that $T_pG = S(T_pM)$ and if M' is any leaf of the integrable distribution TG, then, for any $X \in \Gamma(TM')$, $g(\nabla'^{\perp}_X H, E) = 0$ and $g(\nabla'^{\perp}_X H, N) = 0$ lead to H is a constant vector field on M'.

6. Concluding Remarks

It is well known that the geometry of a lightlike hypersurface depends on the chosen screen distribution. So, it is important to investigate the relationship between some geometrical objects, studied above, with the change of the screen distribution. Note that the local second fundamental form B of M on $\mathcal{U}$ is independent of the choice of the screen distribution (see [9]).

Now, we study the effect of the change of the screen distribution on those above results which also depend on other geometric objects apart from B. Recall the following four local transformation equations (see [9] page 87) of a change from $S(TM)$ to another screen $\widetilde{S(TM)}$:

$$\begin{aligned}
\tilde{K}_i &= \sum_{j=1}^{2n-1} K_i^j (K_j - \epsilon_j c_j E), \\
\tilde{N} &= N - \frac{1}{2}(\sum_{i=1}^{2n-1} \epsilon_i (c_i)^2) E + \sum_{i=1}^{2n-1} c_i K_i, && (6.1)\\
\tilde{\tau}(X) &= \tau(X) + B(X, N' - N), \\
\tilde{\nabla}_X Y &= \nabla_X Y + B(X,Y)(\frac{1}{2}(\sum_{i=1}^{2n-1} \epsilon_i (c_i)^2) E - \sum_{i=1}^{2n-1} c_i K_i), && (6.2)
\end{aligned}$$

where $\{K_i\}$ and $\{\tilde{K}_i\}$ are the local orthonormal basis of $S(TM)$ and $\widetilde{S(TM)}$ with respective transversal sections N and $\tilde{N}$ for the same null section E. Here, c_i and K_i^j are smooth functions on $\mathcal{U}$ and $\{\epsilon_1, ..., \epsilon_{2n-1}\}$ is the signature of the base

$\{K_1, ..., K_{2n-1}\}$. Denote by κ the dual 1-form of $K = \sum_{i=1}^{2n-1} c_i K_i$ (characteristic vector field of the screen change) with respect to the induced metric g of M, that is $\kappa(X) = g(X, K), \quad \forall X \in \Gamma(TM)$.

Let P and $\widetilde{P}$ be projections of TM on $S(TM)$ and $\widetilde{S(TM)}$, respectively with respect to the orthogonal decomposition of TM. So, any vector field X on M can be written as

$$X = PX + \theta(X)E = \widetilde{P}X + \widetilde{\theta}(X)E,$$

where $\theta(X) = \overline{g}(X, N)$ and $\widetilde{\theta}(X) = \overline{g}(X, \widetilde{N})$. Then, using (6.1) we have

$$\widetilde{P}X = PX - \kappa(X)E \quad \text{and} \quad \widetilde{C}(X, \widetilde{P}Y) = \widetilde{C}(X, PY), \quad \forall X, Y \in \Gamma(TM).$$

Using (6.1) and (6.2) the relationship between the second fundamental forms C and $\widetilde{C}$ of the screen distribution $S(TM)$ and $\widetilde{S(TM)}$, respectively, is given by

$$\widetilde{C}(X, PY) = C(X, PY) - \frac{1}{2}\kappa(\nabla_X PY + B(X, Y)K).$$

All equations above depending only on the local second fundamental form C (making equations non unique) are independent of the screen distribution $S(TM)$ if and only if $\kappa(\nabla_X PY + B(X, Y)K) = 0, \ \forall X, Y \in \Gamma(TM)$.

The equations (3.9), (3.19) and (4.13) also are not unique as they depend on C, θ and τ which depend on the choice of a screen vector bundle. The Lie derivatives $L_{(\cdot)}$ and $\widetilde{L}_{(\cdot)}$ of the screen distributions $S(TM)$ and $\widetilde{S(TM)}$, respectively, are related through the relations:

$$\begin{aligned}
(\widetilde{L}_V g)(X, Y) &= (L_V g)(X, Y) - u_X(Y), \\
(\widetilde{L}_{\widetilde{U}} g)(X, Y) &= (L_U g)(X, Y) + \frac{1}{2}\kappa(\nabla_{\{X} P\phi Y\} \\
&\quad + (B(X, \phi Y) + B(\phi X, Y))K) - \frac{1}{2}\kappa(2(\phi Y + X_{\eta(Y)}) \\
&\quad + (u_X(Y) + u(X_{\tau(Y)}))K) + v_X(Y), \\
(\widetilde{L}_{\widetilde{W}} g)(X, Y) &= (L_W g)(X, Y) + (\widetilde{v}(\widetilde{W}) - v(W))\left((L_V g)(X, Y) + u(X_{\tau(Y)})\right) \\
&\quad + (u(\widetilde{W}) - u(W))\left((L_U g)(X, Y) - v(X_{\tau(Y)})\right) \\
&\quad - u(\widetilde{W})\kappa(X_{\eta(Y)}),
\end{aligned}$$

where

$$f_X(Y) = f(X)B(Y, K) + f(Y)B(X, K), \ \nabla_{\{X} P\phi Y\} = \nabla_X P\phi Y + \nabla_Y P\phi X,$$
$$\widetilde{v}(X) = v(X) - \frac{1}{2}\kappa(\phi X + u(X)K), \ X_{f(Y)} = Xf(Y) + Yf(X),$$

f denoting a 1-form.

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