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Research Article

On size function topology and fixed point theorems

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Abstract
In this paper, we introduce and study a metric-like (non-necessarily metric) topology that is weaker than the original topology of a given topological space. The results are used to provide more useful and more general versions of some of the classical fixed point theorems.

Keywords: Banach Contraction Principle, Cantor Intersection Theorem, Fixed point theorems, Fixed set theorems, Metric, Topological spaces.

1 Introduction

The Banach contraction principle states that the sequence of iterates of any contraction on a metric space converges to a unique fixed point. Such a theorem has found numerous important applications for over half a decade. It is not surprising that there have been several attempts to extend such a theorem to more general settings and in various directions. One branch of generalizations is based on the replacement of the contractivity condition imposed on the function (see e.g. [6, 13]). Matkowski's fixed point theorem is one such extension. Another approach is to alter the metric structure of the space in consideration. For example, the study has been extended to the settings of ν -generalized metric spaces (see e.g. [1, 2]), b -metric (see e.g. [3, 6, 8, 9, 13, 14]), semimetric spaces (see e.g. [5, 11]), pseudometric (or generalized quasi-pseudometric) spaces (see e.g. [15]), and quasimetric spaces (see e.g. [4]). One can arguably say that most of new results on the extension of the Banach contraction principle are obtained more or less from some kind of combinations of the above two approaches. The common feature of most of these assorted extension methods is that they all seem to imply that the proofs of the fixed point theorems do not require the entire force of metric properties. This fact indicates the existence of a more general setting of which these different methods are all special cases.
As in many areas of mathematics, it is always desirable and useful to have at our disposal a theory at a level of generality that will allow a wide of a spectrum of applications as possible. The object of this paper is to derive, unify, extend and generalize some results concerning metric topological properties. The generalization, although unabashedly derived from ideas of the classical metric spaces, has the virtue of subsuming, and exposing to a different perspective, some of the general properties of topological spaces and

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their applications. The utility of such a generalization is illustrated via the study of the fixed point and fixed set theorems.

2 Size function topology

Definition 2.1. *Topological size-function.*
Let (X,τ) be a topological space. By a topological size-function we mean a set function $\delta:\tau \to [0,+\infty]$ that satisfies

1. $\delta(U) = 0$ *if* $U = \emptyset$
2. $\max\{\delta(U),\delta(V)\} \leq \delta(U \cup V) \leq \delta(U) + \delta(V)$ *for all* $U, V \in \tau$.

For example, any Borel measure on the σ -algebra of Borel sets generated by the topology τ, when restricted to τ is a prime example of a topological size-function. If (X, d) is a metric space, the diameter function defined by diam $(A) = \sup\{ d(x,y): x, y \in A \}$ defines a topological size function of the metric topology of *X*. For short, we shall write "*δ is a τ -size-function*" to indicate that "*δ is a topological size function defined on X*".
We now generalize the notion of open balls, which is the building block for metric spaces, to the more general setting of topological spaces.

Definition 2.2. *δ-balls.*
Let (X,τ) be a topological space and δ a τ-size function. If $a \in X$ and $r > 0$, a set of the form

$$B_\delta(a,r) = \bigcup \{ U: a \in U \in \tau, \delta(U) < r \}$$

*is called δ **-ball centered at** a, with δ-radius r.*

It is easy to check that the set of δ-balls constitutes a bases for a new topology on X, that we shall denote by τ_δ. It should be clear that every δ-ball is an element of the topology τ of X. Hence, we have $\tau_\delta \subset \tau$.
The following notion will be crucial in what follows.

Definition 2.3. *k-uniformity.*
Let (X,τ) be a topological space. We say that a τ-size-function δ is k-uniform if $\delta(B_\delta(a,r)) \leq kr$ for some $k > 1$ such that for every $a \in X$ and for every $r > 0$.

For example, it is clear that the diameter function on a Euclidean space is 2 -uniform.
We shall now outline some of the basic concepts and notations that are pertinent to the study of fixed point theorem. For the notion of convergence and limit, we adopt the definition as devised by E.H. Moore and H.L. Smith (see e.g. [12]) in the context of the topology τ_δ. Recall that a *net of elements of* $E \subset X$ is a directed family $\{x_\alpha \in E: \alpha \in (\Omega, \succ)\}$ where Ω is a nonempty set, and $\succ$ is a binary relation defined on Ω and satisfying

1. If $\omega, \omega', \omega'' \in \Omega$ are such that both $\omega \succ \omega'$ and $\omega' \succ \omega''$, then $\omega \succ \omega''$;
2. If $\omega, \omega'' \in \Omega$ then there exists $\omega'' \in \Omega$ such that $\omega, \omega' \succ \omega''$.

A net $\{x_\alpha \in E: \alpha \in (\Omega, \succ)\}$ is completely determined by the giving of the function $f:(\Omega,\succ) \to E$ defined by $f(\alpha) = x_\alpha$. In what follows, we shall indifferently think of a net as either a family $\{x_\alpha \in E: \alpha \in (\Omega, \succ)\}$ or a function $f:(\Omega, \succ) \to E$.

Definition 2.4. *δ-convergence, δ-limit.*
Let (X,τ) be a topological space and δ a τ-size-function on X. A net $f:(\Omega,\succ) \to E$ is said to δ-converge to a point $a \in X$, if for every $\varepsilon > 0$, there exists $\omega_0 \in \Omega$ such that if $\omega \succ \omega_0$, then $f(\omega) \in B_\delta(a,\varepsilon)$. The point a is called a δ-limit of the net f and shall be denoted by $a = \delta\text{-}\lim\limits_{\succ} f$.

The notion of separation of points is also of importance in this context. A τ-size-function δ on X is said *to separate points of* X, or simply *to be separating*, if for every pair (x,y) of distinct points in X, there exist $r_x, r_y > 0$ such that $B_\delta(x,r_x) \cap B_\delta(y,r_y) = \emptyset$. It should be clear that if the original topology τ of X is Hausdorff then any τ-size function separates points of X.

Proposition 2.1. *Let (X,τ) be a topological space and δ a τ-size-function on X. If a net $f:(\Omega,\succ) \to E$ of elements of subset E of X is δ-convergent then the limit is unique.*
Proof. Suppose to the contrary that $x \neq y$ are both δ-limits of the net $f:(\Omega,\succ) \to E$. Then there exist $r_x, r_y > 0$ such that $B_\delta(x,r_x) \cap B_\delta(y,r_y) = \emptyset$. Let $r = \min\{r_x, r_y\}$. Since $x = \delta\text{-}\lim\limits_{\succ} f$ (resp. $y = \delta\text{-}\lim\limits_{\succ} f$), there exits $\omega_1 \in \Omega$ (resp. $\omega_2 \in \Omega$) such that if $\omega \succ \omega_1$ (resp. $\omega \succ \omega_2$), $f(\omega) \in B_\delta(x,r)$ (resp. $f(\omega) \in B_\delta(y,r)$). It follows that for $\omega \succ \omega_1, \omega_2$, we have $f(\omega) \in B_\delta(x,r) \cap B_\delta(y,r) = \emptyset$. This contradiction proves the proposition.

Remark 2.1. *It is worth noticing that the above* Definition 2.4 *does not require the δ-limit of a δ-convergent net of elements of E to be an elements of E.*

Definition 2.5. *δ-completeness.*
Let (X,τ) be a topological space and δ a τ-size-function on X. Then a subset E is said to be δ-complete if every δ-convergent net $f:(\Omega,\succ) \to E$ δ-converges to a δ-limit in E.

For the particular case $\Omega = \mathbb{N}$ and $\succ = >$, we say that "*$f:(\Omega,\succ) \to E$ is a δ-Cauchy sequence*" in place of "*$f:(\Omega,\succ) \to E$ is a δ-Cauchy net*". We shall call the set of all δ-limits of nets of elements of E the *δ-completion* of the set E. Such a set will be denoted by $\bar{E}^\delta$. That is to say, $a \in \bar{E}^\delta$ if and only if there exists a net $f:(\Omega,\succ) \to E$ i such that $a = \delta\text{-}\lim\limits_{\succ} f$. We notice that every element $a \in E$ is the δ-limit of the constant net $f(\omega) = a$, for all $\omega \in \Omega$. Therefore, we always have $E \subset \bar{E}^\delta$. Hence Definition 2.5 can be rephrased as follows: *"A subset E of a topological space X is δ-complete if $E = \bar{E}^\delta$".*

Proposition 2.2. *Let (X,τ) be a topological space and δ a τ-size-function on X. Let $f:(\Omega,\succ) \to E$ be a δ-convergent net of elements of a subset E of X. Then for every $\varepsilon > 0$, there exists $\omega_0 \in \Omega$ such that $f(x) \in B_\delta(f(y),\varepsilon)$ whenever $x,y \succ \omega_0$.*

Proof. Suppose $a = \delta\text{-}\lim\limits_{\succ} f$. Given $r > 0$, there is $\omega_0 \in \Omega$ such that for $x,y \succ \omega_0$, we have $f(x), f(y) \in B_\delta(a,r/2)$. Thus there exist $A \supset \{a, f(x)\}$, $\delta(A) < r/2$ and $B \supset \{f(y), a\}$, $\delta(B) < r/2$. It follows that $\{f(x), f(y)\} \subset A \cup B$, $\delta(A \cup B) \leq \delta(A) + \delta(B) < r$, and therefore, $f(x) \in B_\delta(f(y),r)$ as desired.
A net $f:(\Omega,\succ) \to E$ of elements of E is said to be *δ-Cauchy* if it satisfies the conclusion of Proposition 2.2. Thus, Proposition 2.2 states that every δ-convergent net $f:(\Omega,\succ) \to E$ is δ-Cauchy. We shall see (Theorem 2.1) that the converse turns out to be true under the condition that every δ-Cauchy sequence in E δ-converges to an element in E. This later condition shall be referred to as the *δ-sequential completeness* of the set E.
Recall that given two directed sets $(\Gamma,\succcurlyeq)$ and $(\Omega,\succ)$, if there exists a function $\varphi:\Gamma \to \Omega$ with the property that for each $\omega_0 \in \Omega$, there exists $\gamma_0 \in \Gamma$ such that whenever $\gamma \succcurlyeq \gamma_0$ then $\varphi(\gamma) \succ \omega_0$. Then we say that the

net $f \circ \varphi: (\Gamma, \succcurlyeq) \rightarrow E$ is a *subnet* of the net $f: (\Omega, \succ) \rightarrow E$. The function $\varphi: \Gamma \rightarrow \Omega$ will be called a *redirecting function.*

Proposition 2.3. *Let (X, τ) be a topological space and δ a separating τ-size-function on X. Then a net $f: (\Omega, \succ) \rightarrow E$ of elements of a subset E of δ-converges to an element of X if, and only if each of its subnets δ-converges to the same element.*

Proof. Clearly, the condition is sufficient. For the necessity, let $f: (\Omega, \succ) \rightarrow E$ be δ -convergent and assume that $a = \delta\text{-}\lim\limits_{\succ} f$. Let $\varphi: \Gamma \rightarrow \Omega$ be a redirecting function. Given $r > 0$, there exists $\omega_0 \in \Omega$ such that $f(x) \in B_\delta(a, r)$ whenever $x \succ \omega_0$, Let $\gamma_0 \in \Gamma$ be such that whenever $\gamma \succcurlyeq \gamma_0$ then then $\varphi(\gamma) \succ \omega_0$. Then $f(\varphi(\gamma)) \in B_\delta(a, r)$ whenever $\gamma \succcurlyeq \gamma_0$. This shows that $f \circ \varphi$ is δ-convergent and that $a = \delta\text{-}\lim\limits_{\succ} f \circ \varphi$.
We also have the following:

Proposition 2.4. *Let (X, τ) be a topological space and δ a separating τ-size-function on X. Let E be a subset of X and $f: (\Omega, \succ) \rightarrow E$ a δ-Cauchy net. If one of the subnets of f δ-converges to some element a, then f δ-converges to a.*

Proof. Assume that $f: (\Omega, \succ) \rightarrow E$ is δ-Cauchy net and assume that $f \circ \varphi$ is δ-convergent and that $a = \delta\text{-}\lim\limits_{\succ} f \circ \varphi$ for some redirecting function $\varphi: \Gamma \rightarrow \Omega$. Given $r > 0$, let $\omega_0 \in \Omega$ be such that whenever $x, y \succ \omega_0$, $f(x) \in B_\delta\left(f(y), \frac{r}{2}\right)$; that is, there exists $A \in \tau$, $A \ni f(x), f(y)$, and $\delta(A) < r/2$ whenever $x, y \succ \omega_0$. Chose $\gamma_0 \in \Gamma$ such that whenever $\gamma \succcurlyeq \gamma_0$ then $\varphi(\gamma) \succ \omega_0$ and $f(\varphi(\gamma)) \in B_\delta\left(a, \frac{r}{2}\right)$; that is, there exists $B \in \tau, B \ni a, f(\varphi(\gamma))$ and $\delta(B) < r/2$. It follows that whenever $\omega \succcurlyeq \omega_0$, we have $A \cup B \ni f(x), a$, and $\delta(A) \leq \delta(A \cup B) + \delta(B) < r$. This shows that f is δ-convergent and that $a = \delta\text{-}\lim\limits_{\succ} f$.

Theorem 2.1. *δ-completeness theorem***.**
Let (X, τ) be a topological space and δ a separating τ-size-function on X. Let E be a subset of X. Then E is δ-complete if and only if it is δ-sequentially complete.

Proof. We only need to prove the sufficiency. Let E be δ-sequentially complete and let $f: (\Omega, \succ) \rightarrow E$ be δ-Cauchy. Given $r > 0$, there exists $\omega_0 \in \Omega$ such that whenever $x, y \succ \omega_0$, $f(x) \in B_\delta(f(y), r)$. We choose successively $\omega_1, \omega_2, \ldots \in \Omega$ such that $\omega_n \succ \omega_{n-1}$ and $f(x) \in B_\delta(f(y), 1/n)$ whenever $x, y \succ \omega_n$. Then the sequence $n \mapsto f(\omega_n)$ is a δ-Cauchy subnet of f, so it δ-converges to some δ-limit. The result follows from Proposition 2.4.
We now turn our attention to another important property: the notion of δ-cmpactness.

Theorem 2.2. *δ-compactness theorem***.**
Let (X, τ) be a topological space and δ a separating τ-size-function on X.A subset A of X has all the following properties or else it has none of them:

1. *From every covering of A by δ-open sets it is possible to extract finite sets which cover A.*
2. *Every net of elements of A has a δ-cluster point in A.*
3. *Every net of elements of A contains a subnet δ -converging to a point in A.*

Proof. Suppose A has property *1*, and let $f: (\Omega, \succ) \rightarrow A$ be a net. If f had no δ-cluster point in A, for each $a \in A$, we could find ε_a such that $f(\alpha)$ is eventually out of $B_\delta(a, \varepsilon_a)$. Finitely many of these δ-balls cover A, say

$$A \subset \bigcup_{i=1}^{n} B_\delta(a_i, \varepsilon_{a_i}).$$

For each i, let α_i be such that $f(\alpha) \notin B_\delta(a_i, \varepsilon_{a_i})$ whenever $\alpha \succ \alpha_i$. It follows that if $\alpha \succ \alpha_i, i = 1,2,..,n$ then

$$f(\alpha) \notin \bigcup_{i=1}^{n} B_\delta(a_i, \varepsilon_{a_i})$$

yet $f(\alpha)$ A. A contradiction! Thus A has property *2*.

Conversely, assume now that A fails property *1*. Then there is a collection $\mathcal{U}$ of δ -open sets covering A but such that no finite subcollection of $\mathcal{U}$ covers A. Let $2^{[\mathcal{U}]}$ denote the subset of the power set of $\mathcal{U}$ consisting of all finite subcollections of $\mathcal{U}$. It is easy to see that the relation defined by $\mathcal{F} \succ \mathcal{E}$ if each set that belongs to $\mathcal{E}$ also belongs to $\mathcal{F}$, is a direction on $2^{[\mathcal{U}]}$. By our hypothesis, for each element $\mathcal{E} \in 2^{[\mathcal{U}]}$, there exists an element of A, which we shall call $f(\mathcal{E})$, that is not in any of sets in E. Hence, we have defined a net $f: (2^{[\mathcal{U}]}, \succ) \rightarrow A$. For each $a \in A$, pick $U \in \mathcal{U}$ that contains a. Consider the singleton $\{U\} \in 2^{[\mathcal{U}]}$. Thus if $\mathcal{E} \in 2^{[\mathcal{U}]}$, and $\mathcal{E} \succ \{U\}$ then $f(\mathcal{E})$ is not in any of sets in $\mathcal{E}$. In particular, $f(\mathcal{E}) \notin U$. Thus a is not a δ-cluster point for A. This holds for all $a \in A$, so A lacks property *2*.

Clearly, property *3*. implies property *2*. To show the converse, let $f: (\Omega, \succ) \rightarrow A$ be a net with some cluster point $a \in A$. Let $\mathcal{B}$ the set of all pairs $(B_\delta(a, \varepsilon), \alpha)$ where $\varepsilon > 0$ and $\alpha \in \Omega$ such that $f(\alpha) \in B_\delta(a, \varepsilon)$. It is easy to check that the relation defined by $(B_\delta(a, \varepsilon'), \alpha') \succcurlyeq (B_\delta(a, \varepsilon), \alpha)$ if $\varepsilon' \leq \varepsilon$ and $\alpha' \succ \alpha$, is a direction on $\mathcal{B}$. Consider the function $\varphi: (\mathcal{B}, \succcurlyeq) \rightarrow (\Omega, \succ)$ defined by $\varphi((B_\delta(a, \varepsilon), \alpha)) = \alpha$. Let $\alpha \in \Omega$ and $\varepsilon > 0$. Then there exists $\alpha' \succ \alpha$ such that $f(\alpha') \in B_\delta(a, \varepsilon)$. Consider $(B_\delta(a, \varepsilon), \alpha')$. Then whenever $(B_\delta(a, \varepsilon'), \gamma) \succcurlyeq (B_\delta(a, \varepsilon), \alpha')$, we have $\gamma \succ \alpha' \succ \alpha$, i.e. $\varphi((B_\delta(a, \varepsilon'), \gamma)) \succ \alpha$. Hence, $f \circ \varphi: (\mathcal{B}, \succcurlyeq) \rightarrow A$ is indeed a subnet of $f: (\Omega, \succ) \rightarrow A$. On the other hand, since whenever $(B_\delta(a, \varepsilon'), \gamma) \succcurlyeq (B_\delta(a, \varepsilon), \alpha')$, we have $f(\gamma) \in B_\delta(a, \varepsilon)$. Hence, $a = \delta\text{-}\lim_{\succcurlyeq} f \circ \varphi$. The proof is complete.

A subset of X satisfying any of the conditions of Theorem 2.2 is said to be δ-*compact*. We say that a subset E of X is called δ-*totally bounded* if, for every $\varepsilon > 0$, there exists a covering of E by finitely many δ-balls of radius ε.

Theorem 2.3. δ-*total boundedness theorem***.**
Let X be a topological space X and let δ be a k-uniform τ -size-function. Then the following statements are equivalent for a subset E of X

1. *E is δ -totally bounded.*
2. *Every sequence of elements of E admits a Cauchy subsequence*.

Proof. 1. ⇒ 2. Assume that E is δ-totally bounded and let $\{x_n\}$ be a sequence in E. Consider a sequence $\{\varepsilon_n\}$ of positive real numbers converging to 0.

For ε_1, there exists $\{a_{1_1}, a_{1_2}, \dots, a_{1_{J_1}}\}$ such that $E_1 \subset E \subset \bigcup_{i=1}^{J_1} B_\delta(a_1, \varepsilon_1)$. Thus at least one of the δ-balls $B_\delta(a_1, \varepsilon_1)$ contains infinitely many terms of $\{x_n\}$. Let B_1 denotes one of them, and let E_1 be the part of $\{x_n\}$ contained in B_1. Pick $x_{n_1} \in E_1$.

For ε_2, there exists $\{a_{2_1}, a_{2_2}, \dots, a_{2_{J_2}}\}$ such that $E \subset \bigcup_{i=1}^{J_2} B_\delta(a_2, \varepsilon_2)$. Thus at least one of the δ-balls $B_\delta(a_2, \varepsilon_2)$ contains infinitely many terms of E_1. Let B_2 denotes one of them, and let E_2 be the part of E_1 contained in B_1. Pick $x_{n_2} \in E_2$ where $n_2 > n_1$.

Continuing this process, we obtain a nested sequence of δ-balls $\{B_j\}$ of radius ε_j and a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \in B_{j1}$ and that $x_{n_l} \in B_j$ whenever $l > j$. It follows that $x_{n_{l+p}} \in B_\delta(x_{n_l}, k\varepsilon_j)$. This shows that $\{x_{n_j}\}$ is Cauchy. Hence 1. ⇒ 2.

2. ⇒ 1. Assume that 2. holds. Fix $\varepsilon > 0$. Choose $x_1 \in E$. If it happens that every element of E is in $B_\delta(x_1, \varepsilon)$ then we are done. If not, choose $x_2 \in E \setminus B_\delta(x_1, \varepsilon)$. If it happens that every element of E is in $B_\delta(x_1, \varepsilon) \cup B_\delta(x_2, \varepsilon)$, then we are done. We repeat this process. We need to prove that this process must stop after a finite number of steps. Assume to the contrary that this is not the case. Then we would have a sequence $\{x_n\} \subset E$ such that $x_n \notin B_\delta(x_m, \varepsilon)$ for $n \neq m$. One cannot extract a Cauchy subsequence of such a sequence. Contradiction. Thus 2. ⇒ 1; The proof is complete.

The *Cantor's Intersection Theorem* states that a decreasing nested net of non-empty compact subsets of a topological space has nonempty intersection. Our next result is a slight generalization of such a theorem. If (X, τ) is a topological space and δ a τ-size-function on X, we shall extend δ to the whole power set 2^X by simply setting $\delta^*(A) = \sup\{\delta(U): U \in \tau, U \subset A\}$.

Theorem 2.4. *Extended Cantor's Intersection Theorem***.**
Assume that X be a δ-complete. Let $\{C_\alpha: \alpha \in (\Omega, \succ)\}$ be a net of δ-complete nested nonempty subsets of X

1. *If $\lim_\alpha \delta^*(C_\alpha) = 0$, then the intersection $\bigcap_\alpha C_\alpha$ contains at least one point.*
2. *If in addition, the τ-size function δ is separating, then the intersection $\bigcap_\alpha C_\alpha$ contains exactly one point.*

Proof. For each $\alpha \in \Omega$, pick $x_\alpha \in C_\alpha$. We have $x_\gamma \in C_\alpha$ for all $\gamma \in \Omega, \gamma \succ \alpha$. Thus $x_\gamma \in B_\delta(x_\alpha, \delta^*(C_\alpha))$ whenever $\gamma \succ \alpha$. The condition that $\lim_\alpha \delta^*(C_\alpha) = 0$ then implies that the net $\{C_\alpha: \alpha \in (\Omega, \succ)\}$ is δ-Cauchy. Since C_α is δ-complete, this net is δ-convergent to some point $x \in C_\alpha$. This holds for all $\alpha \in \Omega$, thus $x \in \bigcap_\alpha C_\alpha$.
Now assume that δ is separating and suppose to the contrary that the intersection $\bigcap_\alpha C_\alpha$ contains another point $y \neq x$. Then there exists $\varepsilon > 0$ such that $B_\delta(x, \varepsilon) \cap B_\delta(y, \varepsilon) = \emptyset$. In particular, $x \notin B_\delta(y, \varepsilon)$. On the other hand, since $\lim_\alpha \delta^*(C_\alpha) = 0$, we can find $\alpha_0 \in \Omega$, such that whenever $\alpha \succ \alpha_0$ then $\delta^*(C_\alpha) < \varepsilon$. Since both x and y are in C_α, it follows that $x \in B_\delta(y, \delta^*(C_\alpha)) \subset B_\delta(y, \varepsilon)$. Contradiction! The proof is complete.

3 Fixed point theorems

In this section, we revisit some of the classical fixed point theorems. Recall that a fixed point for a mapping $T: X \to X$ is a point $a \in X$ such that $T(a) = a$. In what follows, we shall use the following common standard notation for the n-th iteration of a mapping $f: E \to E$ as follows

$$f^n(x) = f(f(\dots(f(x))))$$

for every $x \in E$.

Let us agree to say that a function $\varphi: [0, \infty) \to [0, \infty)$ is *contractant* if it is increasing and if $\lim_{n\to\infty} \varphi^n(t) = 0$, for all $t > 0$. An example of a contractant function is given by the function $t \to qt$ where $0 < q < 1$. Note that every contractant function $\varphi: [0, \infty) \to [0, \infty)$ has the property that $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$.
The following definition generalizes the notion of a contraction.

Definition 3.1. *δ-contraction.*
Let (X,τ) be a topological space and δ a τ-size-function. A mapping $T: X \to X$ is said to be a δ-contraction if there exists a contractant function $\varphi: [0,\infty) \to [0,\infty)$ such that $\delta^*\left(\overline{T(U)}^{\delta}\right) < \varphi\big(\delta(U)\big)$ for all $U \in \tau$.

We notice that in the context of the topology τ_δ, a mapping $T: X \to X$ is δ-continuous at a point $a \in X$, if for every $\varepsilon > 0$, there exists $r > 0$ such that $T(x) \in B_\delta(T(a), \varepsilon)$ when ever $x \in B_\delta(a, r)$.

Proposition 3.1. *Let (X,τ) be a topological space and δ a k-uniform τ-size-function. If $T: X \to X$ is a δ-contraction mapping, then it is necessarily δ-continuous.*

Proof. Let $a \in X$. Fix $\varepsilon > 0$, and let $x \in B_\delta(a, k^{-1}\varepsilon)$. Then $T(x)$ and $T(a)$ are both in $T\big(B_\delta(a, k^{-1}\varepsilon)\big)$. On the other hand, we have

$$\delta^*\left(T\big(B_\delta(T(a), k^{-1}\,\varepsilon)\big)\right) \le \delta^*\left(\overline{T\big(B_\delta(T(a), k^{-1}\,\varepsilon)\big)}^{\delta}\right)$$
$$\le \varphi\left(\delta\left(T\big(B_\delta(T(a), k^{-1}\,\varepsilon)\big)\right)\right) < \varphi(\varepsilon) < \varepsilon.$$

It follows that $T(x) \in B_\delta(T(a), \varepsilon)$. This proves the δ-continuity of φ at a, and completes the proof.

We are now ready to state and prove an extended version of the *Matkowski's fixed point theorem*.

Theorem 3.1. *Extended Matkowski's fixed point theorem.*
Let (X,τ) be a topological space and δ a τ-size-function. Let $T: X \to X$ be a δ-contraction. If there exists a δ-complete subset A of X such that $\delta^*(A) < \infty$ and $T(A) \subset A$, then

1. *For every $x \in A$, $a = \lim_{n\to\infty} T^n(x)$ is a fixed point for T.*
2. *If in addition, the τ-size function δ is separating and k-uniform, then T admits a unique fixed point.*

Proof. Let $\varphi: [0,\infty) \to [0,\infty)$ be a contractant function for $T: X \to X$. Choose an arbitrary $x \in A$ and define a sequence by setting $x_n = T^n(x)$.
1. We claim that the sequence $\{x_n\}$ is δ-Cauchy.
We let $\delta_0 = \delta^*(A)$. Then $T(x)$ and $T^2(x)$ are both in $T(A)$ and $\delta^*\big(T(A)\big) \le \varphi(\delta_0)$. Iteratively, we have for each $n \in \mathbb{N}$, $T^n(x)$ and $T^{n+1}(x)$ are both in $T^n(A)$ and $\delta^*\big(T^n(A)\big) \le \varphi^n(\delta_0)$. Since $\lim_{n\to\infty} \varphi^n(\delta_0)$, given $\varepsilon > 0$, we can choose n_1 large enough so that for $p \in \mathbb{N}$, we have $\delta^*\big(T^{n_1+P}(A)\big) < \varepsilon - \varphi(\varepsilon)$. Then we choose $n_2 > n_1$ such that $\varphi^{n_2}(\delta_0) \le \varphi(\varepsilon)$. Then for $n > n_2$ and for every $p \in \mathbb{N}$, $T^n(x)$ and $T^{n+p}(x)$ are in $T^n(A) \cup T^{n+p}(A)$. Since

$$\delta^*\big(T^n(A) \cup T^{n+p}(A)\big) \le \delta^*\big(T^n(A)\big) + \delta^*\big(T^{n+p}(A)\big)$$
$$\le \varphi^n(\delta_0) + \varepsilon - \varphi(\varepsilon) < \varepsilon,$$

it follows that for $n > n_2$ and for every $p \in \mathbb{N}$, $T^{n+p}(x) \in B_\delta(T^n(x), \varepsilon)$. Hence, our claim.
Since A is δ -complete, $\{x_n\}$ converges to some $a \in A$. The continuity of T then implies that $a = Ta$.
2. Suppose that the τ-size function δ is separating and k-uniform. Assume that there exists b such that $b = T(b)$ and $b \ne a$. Then there exists $\varepsilon > 0$ such that $B_\delta(a,\varepsilon) \cap B_\delta(b,\varepsilon) = \emptyset$. In particular, $b \notin B_\delta(a,\varepsilon)$. On the other hand, we clearly have $T(a)$ and $T(b)$ are in $T\left(B_\delta\left(a, \frac{\varepsilon}{4k}\right) \cup B_\delta\left(b, \frac{\varepsilon}{4k}\right)\right)$. Since

$$\delta^*\left(T\left(B_\delta\left(a, \frac{\varepsilon}{4k}\right) \cup B_\delta\left(b, \frac{\varepsilon}{4k}\right)\right)\right) \le \varphi\left(\delta\left(B_\delta\left(a, \frac{\varepsilon}{4k}\right) \cup B_\delta\left(b, \frac{\varepsilon}{4k}\right)\right)\right)$$
$$\le \varphi\left(\delta\left(B_\delta\left(a, \frac{\varepsilon}{4k}\right) \cup\right) + \delta\left(B_\delta\left(b, \frac{\varepsilon}{4k}\right)\right)\right)$$

$$\leq \varphi\left(\frac{\varepsilon}{4}+\frac{\varepsilon}{4}\right) < \frac{\varepsilon}{2},$$

we have $b = T(b) \in B_\delta(a, \varepsilon/2)$. Contradiction! The proof is complete.

We say that a mapping $T: X \to X$ is δ-*Lipschitz* if there exists a constant $q > 0$ such that

$$\delta^*\left(\overline{T(E)}^\delta\right) < q\delta^*(E)$$

for every $E \in 2^X$. If $0 < q < 1$, the function $t \mapsto qt$ is contractant for T. As an immediate corollary of the above extension of the Matkowski's fixed point theorem, we have the following extension of the *Banach Fixed Point Theorem*.

Theorem 3.2. *Generalized Banach Fixed Point Theorem***.**
Let (X, τ) be a topological space and δ a τ-size-function. Let $T: X \to X$ be a δ-Lipschitz with constant $q \in$ $(0,1)$. *If there exists a δ-complete subset A of X such that $\delta^*(A) < \infty$ and $T(A) \subset A$, then*

1. *For every $x \in A$, $a = \lim_{n\to\infty} T^n(x)$ is a fixed point for T.*
2. *If in addition, the τ-size function δ is separating and k-uniform, then T admits a unique fixed point.*

Another corollary of Theorem 3.1 is easily obtained as follows.

Theorem 3.3.
Let (X, τ) be a topological space and δ a τ-size-function. Let $T: X \to X$ be a mapping. Assume that there exists a δ-complete subset A of X such that $\delta^*(A) < \infty$ and $T^m: A \to A$ is a δ-contraction for some natural number m. Then T admits a unique fixed point.

Proof. The case $m = 1$ is exactly that of Theorem 3.1. Assume that $m > 1$. The mapping $S = T^m$ satisfies the hypotheses of Theorem 3.1, hence it admits a unique fixed point, say a in A. Then

$$S(T(a)) = T^{m+1}(a) = T(S(a)) = Ta.$$

In other words, $T(a)$ is also a fixed point of S. Since δ is separating, we have $T(a) = a$. That is, a is a fixed point for T. To see that a is unique, assume that $b = T(b)$. Then $S(b) = T^m(b) = b$. That is, b is a fixed point for S and hence $b = a$. The proof is complete.

Our next result is a consequence of the extended Cantor Intersection Theorem 2.3.
Let (X, τ) be a topological space, δ a τ-size-function, and fix a sequence $\{a_n\}$ of positive numbers converging to 0. Given a subset A of X, let us agree to say that a mapping $T: X \to X$ is *nearly δ-Lipschitz with respect to* $\{a_n\}$ on A if for each $n \in \mathbb{N}$ there exists $q_n \geq 0$ such that for every $E \in 2^A$, we have

$$\delta^*\left(\overline{T^n(E)}^\delta\right) < q_n(\delta^*(E) + a_n).$$

The smallest such constant q_n will be denoted by $q(T^n)$.

Theorem 3.4. *Nearly δ-Lipschitz Fixed Point Theorem***.**
Let (X, τ) be a topological space and δ a τ-size-function. Let $T: X \to X$ be a mapping. Assume that there exists a δ-complete subset A of X such that $\delta^*(A) < \infty$, $T(A) \subset A$, and that T is nearly δ-Lipschitz on A with respect to a sequence $\{a_n\}$ of positive numbers converging to 0. *Suppose that* $\limsup_{n\to\infty}[q(T^n)]^{\frac{1}{n}} < 1$.
Then

1. *For every $x \in X$, $a = \lim_{n\to\infty} T^n(x)$ is a fixed point for T.*
2. *If in addition, the τ-size function δ is separating and k-uniform, then T admits a unique fixed point.*

Proof. Let $M = \sup\{a_n : n \in \mathbb{N}\}$. We choose an arbitrary element $x \in X$ and we define a sequence by setting $x_n = T^n(x)$. We then consider a τ-open set U_0 containing both x and Tx, and let $\delta_0 = \delta(U_0)$. Then for each $n \in \mathbb{N}$, we notice that $T^n(x)$ and $T^{n+1}(x)$ are both in $T^n(U_0)$, and we observe that

$$\delta^*\left(\overline{T^n(U_0)}^{\delta}\right) \leq q_n(\delta_0 + a_n) \leq q_n(\delta_0 + M).$$

This implies that $T^{n+1}(x) \in B_\delta(T^n(x), \delta_n)$ where $\delta_n = q_n(\delta_0 + M)$. It follows that for every $p \in \mathbb{N}$,

$$T^{n+p}(x) \in B_\delta\left(T^n(x), \sum_{i=1}^{p} \delta_{n+i}\right).$$

Since $\limsup_{n\to\infty}[q(T^n)]^{1/n} < 1$, the series $\sum q(T^n)$, and hence the series $\sum \delta_n$ converges. This implies that for every p, the sequence $\varepsilon_{n,p} = \sum_{i=1}^{p} \delta_{n+i} \to 0$ as $n \to \infty$.
Now, let $\mathbb{N} \times \mathbb{N}$ be directed as follows: $(n,p) \succ (n',p')$ if $n > n'$ or $n = n'$ and $p > p'$. Let

$$C_{n,p} = \overline{B_\delta\left(T^n(x), \sum_{\iota=1}^{p} \delta_{n+\iota}\right)}^{\delta}.$$

We then observe that $\{C_{n,p} : (n,p) \in \mathbb{N} \times \mathbb{N}\}$ is a net of δ -complete nested subsets of X of positive size with $\lim_{(n,p)} \delta^*(C_{n,p}) = 0$. The extended Cantor Intersection Theorem 2.3 now finishes the proof.

4 Fixed set theorems

Let X be a complete metric space. Denote by $\mathcal{K}(X)$ the space of non-empty compact subsets of X. It is a well-known fact that $\mathcal{K}(X)$ is a complete metric space when endowed with the Hausdorff metric. The Hutchinson's Theorem [10] (see also [7]) states that if $\{K_1, \dots, K_n\}$ is a family of contractions on X with respective Lipschitz constants $\{k_1, \dots, k_n\}$, then the operator K defined on $\mathcal{K}(X)$ by $K(A) = \bigcup_{i=1}^{n} K_i(A)$ is a contraction with Lipschitz constant equal to $k = \max\{k_1, \dots, k_n\}$, The Banach contraction principle then implies the existence of a compact set E such that $K(E) = E$.
Our next result extends such a result.

Theorem 4.1. *Extended Hutchkinson's fixed set theorem.*
Let (X,τ) be a topological space and δ a τ-size-function. Let $\mathcal{K}(X)$ denote the space of non-empty δ-compact subsets of X. Let $K : \mathcal{K}(X) \to \mathcal{K}(X)$ be a monotone mapping, i.e. $K(A) \subset K(B)$ whenever $A \subset B$ in $\mathcal{K}(X)$. If there exists $A \in \mathcal{K}(X)$ such that $K(A) \subset A$, then there exists $B \in \mathcal{K}(X)$, ($B \subset A$) such that $K(B) = B$.

Proof. Let $\mathcal{H}(X)$ be the collection consisting of subsets in of $\mathcal{K}(X)$ satisfying $K(A) \subset A$. By hypothesis, $\mathcal{H}(X)$ is not empty. We partially order $\mathcal{H}(X)$ by inclusion. According to the Hausdorff maximal principle, there exists a maximal linearly ordered subcollection $\{A_i : i \in I\}$. Let $B = \bigcap_{i\in I} A_i$. By the classical Cantor's Intersection Theorem in the context of the τ_δ topology, B is a nonempty δ-compact set. On the other hand, since $K(A_i) \subset A_i$ i for all $i \in I$, we also have $K(B) \subset B$, and hence by monotonicity $K(K(B)) \subset K(B)$. Now by the maximality of $\{A_i : i \in I\}$, we must have $K(B) = B$.

Note that no continuity properties is required in the above Theorem 4.1. A special case is as follows:

Theorem 4.2. *Let (X,τ) be a topological space and δ a τ-size-function. Let $T_i : X \to X$, $i = 1,2,\dots,n$ be a finite collection of δ-continuous mappings. If there exists a δ-compact subset A of X such that $T_i(A) \subset A$ for $i = 1,2,\dots,n$ then there exists a δ-compact subset B of A such that $K(B) = B$.*

Proof. It suffices to notice that the mapping $K: \mathcal{K}(X) \to \mathcal{K}(X)$ defined by $K(A) = \bigcup_{i=1}^{n} T_i(A)$ for every $A \in \mathcal{K}(X)$ satisfies the hypotheses of Theorem 4.1.
In what follows, we let $\mathcal{C}(X)$ denote the space of non-empty δ-complete subsets of X. The following result is a variant of Theorem 4.1.

Theorem 4.3. *A variant of the extended Hutchkinson's fixed set theorem.*
Let (X,τ) be a topological space and δ a τ-size-function. Let $\mathcal{C}(X)$ denote the space of non-empty δ-compact subsets of X. Let $K: \mathcal{C}(X) \to \mathcal{C}(X)$ be a monotone mapping, i.e. $K(A) \subset K(B)$ whenever $A \subset B$ in $\mathcal{C}(X)$. If there exists $A \in \mathcal{C}(X)$ such that $K(A) \subset A$, and $\delta^*(A) < \infty$, then there exists $B \in \mathcal{C}(X)$, ($B \subset A$) such that $K(B) = B$.

For the proof, we need the following two lemmas.

Lemma 4.1. *Let $K: \mathcal{C}(X) \to \mathcal{C}(X)$ be a monotone mapping. If $\{C_\alpha : \alpha \in (\Omega, \succ)\}$ is a net of δ-complete nested nonempty subsets of X such that $\lim_\alpha \delta^*(C_\alpha) = 0$, then $K(\bigcap_\alpha C_\alpha) = \bigcap_\alpha K(C_\alpha)$.*

Proof. It is clear that for every $x \in \bigcap_\alpha C_\alpha$, $K(\{x\}) \in K(C_\alpha)$ for each $\alpha \in \Omega$. Thus
$$K\left(\bigcap_\alpha C_\alpha\right) \subset \bigcap_\alpha K(C_\alpha).$$
Conversely, we still denoted by K the function defined by $K(x) = K(\{x\})$. We then notice that such a function is a contraction on each C_α. If y $\in \bigcap_\alpha K(C_\alpha)$, then for every $\alpha \in \Omega$, $y = K(x_\alpha)$ for some $x_\alpha \in C_\alpha$. Since $\lim_\alpha \delta^*(C_\alpha) = 0$, the extended Cantor's Intersection Theorem 2.3 implies that $\bigcap_\alpha C_\alpha$ is not empty, say $x \in \bigcap_\alpha C_\alpha$. Since the contraction K is continuous on each C_α, we infer that $y = K(x)$. The lemma is proved.

Lemma 4.2. *Let $K: 2^X \to 2^X$ monotonically maps $\mathcal{C}(X)$ into $\mathcal{C}(X)$. If $E \in 2^X$ then $\overline{K(E)}^\delta \subset K(\bar{E}^\delta)$.*

Proof. Since K maps $\mathcal{C}(X)$ into $\mathcal{C}(X)$, we have $K(\bar{E}^\delta)$ is δ-complete, thus $\overline{K(\bar{E}^\delta)}^\delta = K(\bar{E}^\delta)$. On the other hand, it is clear that $\overline{K(E)}^\delta \subset \overline{K(\bar{E}^\delta)}^\delta$. Therefore $\overline{K(E)}^\delta \subset K(\bar{E}^\delta)$.
We are now ready to give the proof of Theorem 4.3.

Proof of Theorem 4.3. Let $A \in \mathcal{C}(X)$ such that $\delta^*(A) < \infty$, and $K(A) \subset A$. Define $C_n = \overline{\bigcup_{k=n}^{\infty} K^k(A)}^\delta$. It is clear that $\{C_n\} \subset \mathcal{C}(X)$. Since $C_n \subset A$, and since K is a δ-contraction, we have
$$\begin{aligned}\delta^*(C_n) &\le \varphi(\delta^*(C_n)) \le \varphi(\delta^*(C_{n-1})) \\ &\le \varphi^2(\delta^*(C_{n-2})) \le \cdots \le \varphi^{n-1}(\delta^*(C_1)) \le \varphi^n(\delta^* A).\end{aligned}$$

Thus $\lim_n \delta^*(C_n) = 0$. By Lemma 4.1, $K(\bigcap_n C_n) = \bigcap_n K(C_n)$. The Cantor Intersection Theorem 2.3 implies that $C = \bigcap_n C_n$ is not empty.
To finish the proof, we show that $K(C) = C$. First since $K(C_n) \subset C_n$ for each n, we have $K(C) \subset C$. On the other hand, it follows from Lemma 4.1 that
$$K(C) = K\left(\bigcap_n \overline{\bigcup_{k=n}^{\infty} K^k(A)}^\delta\right) = \bigcap_n K\left(\overline{\bigcup_{k=n}^{\infty} K^k(A)}^\delta\right).$$

Then by Lemma 4.2, we have

$$K(C) \supset \bigcap_n \overline{K\left(\bigcup_{k=n}^{\infty} K^k(A)\right)}^{\delta} = \bigcap_n \overline{\left(\bigcup_{k=n}^{\infty} K^{k+1}(A)\right)}^{\delta}$$
$$= \bigcap_n \overline{\left(\bigcup_{k=n+1}^{\infty} K^k(A)\right)}^{\delta} = C.$$

The proof is complete.

We finish this paper by showing that the above result can be used to prove the existence of a fixed point in Theorem 3.1.

Alternative proof of Theorem 3.1. Let $\varphi: [0, \infty) \to [0, \infty)$ be a contractant function for $T: X \to X$. Then the mapping $K: \mathcal{C}(X) \to \mathcal{C}(X)$ defined by $K(E) = \{T(x): x \in E\}$ satisfies the hypotheses of Theorem 4.3. Then there exists $C \in \mathcal{C}(X)$, $C \subset A$ such that $K(C) = C$. In fact we have $C = \bigcap_n \overline{\bigcup_{k=n}^{\infty} K^k(A)}^{\delta}$. Since δ is separating and $\lim_n \delta^*(C_n) = 0$, according to the extended Cantor's Intersection Theorem 2.3, C is a singleton, i.e. $C = \{a\}$. Therefore, $K(\{a\}) = \{a\}$, and hence $T(a) = a$. The proof is complete.

5 Conclusion

In this paper, we have introduced the concept of size function topology that naturaly generalizes the metric topology. We were able to use such a novel approach to obtain more useful and generalized forms of some of the classical fixed point theorem.

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