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STRING HOMOLOGY OF A PRODUCT OF SPHERES AND THE WITT ALGEBRA

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Abstract

Let X be a finite product of even dimensional spheres, we show that the string homology of X contains a finite product of copies of the Witt Lie algebra.

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1 Introduction

In this paper, all homology coefficients are taken in the field of rational numbers $\mathbb{Q}$. By the work of Chas and Sullivan [2], the desuspended homology of the free loop space on an n-dimensional manifold M, $\mathbb{H}_*(M^{S^1}) = H_{*-n}(M^{S^1})$ admits a Gerstenhaber structure and in particular a Lie bracket. By Cohen-Jones [3] and Félix-Thomas [6], there is an isomorphism of Gerstenhaber algebras $\mathbb{H}_*(M^{S^1}) \simeq HH^*(C_*(\Omega M), C_*(\Omega M))$. In Félix-Menichi-Thomas [5], for any graded algebra $A = (TV, d)$, $HH^*(A, A)$ can be computed in terms of derivations on A.

Now recall that the rational Witt Lie algebra is the graded Lie algebra $W =< e_i, i \in \mathbb{Z} >$ with the bracket $[e_i, e_j] = (j-i)e_{i+j}$. Denote by W_+ the positive part of W, that is, $W_+ =< e_i, i \geq 1 >$.

Our main Theorem states.

Theorem. *Let M be a product of n even dimensional spheres, then the Lie algebra $HH_*(M^{S^1}) = HH^*(C_*(\Omega M), C_*(\Omega M))$ contains the sub Lie algebra $\oplus_{i=1}^n W_i$, where each W_i is isomorphic to the Witt algebra W_+.*

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2 Hochshild cohomology and derivations

Let (TV, d) denote the tensor algebra TV together with a differential d such that V is the union $V = \cup V(k)$ of an increasing family of subspaces $V(0) \subset V(1) \subset \cdots$ such that $d(V(0)) = 0$ and $d(V(k)) \subset T(V(k-1))$. This is a quasi-free algebra and for any differential graded algebra (A, δ) there is a quasi-isomorphism of differential graded algebras $(T(V), d) \xrightarrow{\simeq} (A, \delta)$ with $(T(V), d)$ a quasi-free algebra [4]. The algebra $(T(V), d)$ is then called a quasi-free model of (A, δ).

Denote by Der A the differential graded Lie algebra of derivations with the commutator bracket $[-,-]$ and the differential $D = [d, -]$. The differential graded Lie algebra $\widetilde{\mathrm{Der}} A = \mathrm{Der} A \oplus sA$ is defined as follows [9];

$$
\begin{aligned}
&D(\alpha + sx) = D(\alpha) + ad_x - sd(x) \quad \text{where} \quad ad_x(y) = xy - (-1)^{|x||y|} yx, \\
&[\alpha, \beta + sx] = [\alpha, \beta] + (-1)^{|\alpha|} s\alpha(x), \\
&[sx, sy] = 0 \quad \text{with} \quad \alpha, \beta \in \mathrm{Der}\, A \text{ and } (sA)_i = A_{i-1}.
\end{aligned}
$$

Let $C^*(A, A)$ denote the Hochshild cochain complex with coefficients in A and $HH^*(A; A)$ the *Hochschild cohomology* of A with coefficients in A. For a supplemented differential graded coalgebra (C, d), $\bar{\Omega} C$ denotes the reduced cobar [4]. The cobar construction permits one to calculate the homology of loop spaces [1], that is, $H_*(\bar{\Omega} C_*(X)) \cong H_*(\Omega X)$.

Proposition 2.1. *[5] Let $(TV, d) \xrightarrow{\simeq} A$ and $(TW, d) = \bar{\Omega} C$ be quasi-free models, then we have isomorphisms of graded Lie algebras*

$$H(\widetilde{\mathrm{Der}}(TW, d)) \cong sHH^*(A; A) \cong H(\widetilde{\mathrm{Der}}(TV, d)),$$

where $A = C^\vee$ is the dual differential graded algebra.

Theorem 2.2. *[5, Theorem 2] Let $A = (TV, d)$ be a quasi-free algebra. Then there exists quasi-isomorphisms of differential graded Lie algebras*

$$s\mathbf{C}^*(A; A) \xleftarrow{\simeq} \widetilde{\mathrm{Der}} A \xrightarrow{\simeq} \mathrm{Der} \widetilde{A},$$

where $\widetilde{A} = (T(V \oplus \mathbb{Q}\varepsilon), \widetilde{d})$ with $|\varepsilon| = 1$, $\widetilde{d}\varepsilon = \varepsilon^2$ and $\widetilde{d}v = dv + \varepsilon v - (-1)^{|v|} v\varepsilon$, $v \in V$.

The theory of $\mathbb{Z}_2$-graded Lie algebras or Lie Superalgebras has been introduced in [8]. Here we recall the following definition.

Definition 2.3. A *Lie Superalgebra* is a $\mathbb{Z}_2$-graded Lie algebra, that is, a $\mathbb{Z}_2$-graded vector space $L = L_{\bar{0}} \oplus L_{\bar{1}}$ where $L_{\bar{0}}, L_{\bar{1}}$ denote the even and odd parts respectively; with a bracket multiplication $[,]$ compatible with the gradation, that is,

$$
\begin{aligned}
&[L_{\bar{0}}, L_{\bar{0}}] \subseteq L_{\bar{0}}, \qquad [L_{\bar{1}}, L_{\bar{1}}] \subseteq L_{\bar{0}}, \\
&[L_{\bar{0}}, L_{\bar{1}}] \subseteq L_{\bar{1}}, \qquad [L_{\bar{1}}, L_{\bar{0}}] \subseteq L_{\bar{1}},
\end{aligned}
$$

and satisfying the following properties.

$$[b, a] = -(-1)^{|a||b|} [a, b] \qquad \text{(Antisymmetry)},$$

$$(-1)^{|a||c|} [a, [b, c]] + (-1)^{|b||a|} [b, [c, a]] + (-1)^{|c||b|} [c, [a, b]] = 0 \quad \text{(Jacobi identity)}.$$

3 Homology of the free loop space of a product of spheres

Let X denote a 1-connected space X. Consider its Quillen model $(\mathbb{L}V,\delta)$ and $(TV,d) = U(\mathbb{L}V,\delta)$, its enveloping algebra. There is a quasi-isomorphism $(TV,d) \xrightarrow{\simeq} C_*(\Omega X,\mathbb{Q})$. Denote by X^{S^1} the free loop space of X, that is, the space of continuous mappings from the circle S^1 to X.

It comes from [3] and Theorems 1, 2 of [5] the following isomorphisms of graded Lie algebras.

$$H_*(X^{S^1}) \cong HH^*(C^*(X),C^*(X)) \cong HH^*(C_*(\Omega X),C_*(\Omega X)) \cong H_*(\widetilde{\mathrm{Der}}(TV,d)).$$

Our aim is to compute $H_*(\widetilde{\mathrm{Der}}(TV,d))$, when (TV,d) is a model of a product of spheres.

Let $X = S^{2n_1} \times \cdots \times S^{2n_k}$. It is a coformal space of which the Quillen minimal model is a bigraded model $(\mathbb{L}(V_0 \oplus V_1 \oplus \cdots \oplus V_{k-1}),\delta)$ such that

$$\varphi : (\mathbb{L}(V_0 \oplus V_1 \oplus \cdots \oplus V_{k-1}),\delta) \xrightarrow{\simeq} \oplus_{i=1}^{k} \mathbb{L}(x_i)$$

is a quasi-isomorphism, x_i is of bidegree $(0,2n_i-1)$ and elements of V_i are of bidegree $(i,*)$. Moreover

(1) φ is of bidegree $(0,0)$

(2) $\delta V_k \subset (\mathbb{L}(V))_{k-1}$

(3) $H_+(\mathbb{L}(V)) = 0$, $H_0(\mathbb{L}(V)) \xrightarrow{\simeq} \oplus_i \mathbb{L}(x_i)$.

We compute the loop space homology in the following case.

Example 3.1. Consider $X = S^2 \times S^2$ and $(T(x,y,z),d)$, where $|x| = |y| = 1, |z| = 3$, $dx = dy = 0, dz = xy + yx$, its (TV,d)-model. Here $V_0 =< x,y >$, $V_1 =< z >$. Consider the even and odd degree derivations

$$\begin{cases} \varphi_m(x) = x^{2m+1} \\ \varphi_m(y) = 0 \\ \varphi_m(z) = \sum_{r+s=2m} x^r z x^s\ , \quad |\varphi_m| = 2m \end{cases}$$
$$\begin{cases} \rho_n(x) = x^{2n+2} \\ \rho_n(y) = 0 \\ \rho_n(z) = \sum_{t+l=2n+1} x^t z x^l\ , \quad |\rho_n| = 2n+1. \end{cases}$$

Observe that

(i) $d(\sum_{r+s=2m} x^r z x^s) = x^{2m+1}y + yx^{2m+1}$,

(ii) $d(\sum_{t+l=2n+1} x^t z x^l) = yx^{2n+2} - x^{2n+2}y$.

Proposition 3.2. *φ_m and ρ_n are non vanishing homology classes of* $\widetilde{\mathrm{Der}}(T(x,y,z))$.

Proof. We will work with φ_m and a similar argument holds for ρ_n. We show first that φ_m is a cycle. Clearly $(D\varphi_m)(x) = 0 = (D\varphi_m)(y)$, so we need only to verify that $D\varphi_m(z) = 0$.

$$\begin{aligned} D\varphi_m(z) &= [d,\varphi_m](z) \\ &= d\varphi_m(z) - \varphi_m(dz) \\ &= d\left(\sum_{r+s=2m} x^r z x^s\right) - \varphi_m(xy+yx) \\ &= x^{2m+1}y + yx^{2m+1} - (\varphi_m(x)y + y\varphi_m(x)) \\ &= x^{2m+1}y + yx^{2m+1} - (x^{2m+1}y + yx^{2m+1}) \\ &= 0. \end{aligned}$$

It remains to show that φ_m is not a boundary. Suppose $\varphi_m = D(\phi + s\alpha)$, where $\phi \in \mathrm{Der}\, T(V)$ and $\alpha \in T(V)$. As $\varphi_m(x) = D\phi(x) + [\alpha, x] = x^{2m+1}$, hence $D\phi(x) = x^{2m+1} - [\alpha, x]$, $d\phi(x) = x^{2m+1} - [\alpha, x]$. This is not possible because $\mathrm{Im}\, d \subset (y)$. Therefore φ_m represents a non vanishing homology class in $H_*(\widetilde{\mathrm{Der}}(T(x,y,z))$. □

Lemma 3.3. *(i)* $\varphi_m\varphi_n = (2n+1)\varphi_{m+n}$ *on* $V = <x,y,z>$.

(ii) $\varphi_m\rho_n = (2n+2)\rho_{m+n}$ *on* $<x,y,z>$.

(iii) $\rho_n\varphi_m = \rho_{m+n}$ *on* $<x,y,z>$.

(iv) $\rho_m\rho_n = 0$.

Proof. We will prove only the first relation as the remaining equalities are verified in a similar way. As $\varphi_k(y) = 0$, we need only to check the equality for x and z. Clearly $\varphi_m\varphi_n(x) = \varphi_m(x^{2n+1}) = (2n+1)x^{2(m+n)+1} = (2n+1)\varphi_{m+n}(x)$.

We now compute $\varphi_m\varphi_n(z)$ for $n \leq m$, computations are similar for $n \geq m$.

$$\begin{aligned} \varphi_m\varphi_n(z) &= \varphi_m(\textstyle\sum_{i=0}^{2n} x^i z x^{2n-i}) \\ &= \textstyle\sum_{i=0}^{2n} i x^{2m+i} z x^{2n-i} + \sum_{i=0}^{2n}(2n-i)x^i z x^{2(m+n)-i} \\ &\quad + \textstyle\sum_{i=0}^{2n} x^i (\sum_{j=0}^{2m} x^j z x^{2m-j}) x^{2n-i} \\ &= x^{2m+1} z x^{2n-1} + 2x^{2m+2} z x^{2n-2} + \cdots + 2n x^{2(m+n)} z \\ &\quad +2n z x^{2(m+n)} + (2n-1) x z x^{2(m+n)-1} + \cdots + x^{2n-1} z x^{2m+1} \\ &\quad + \textstyle\sum_{i=0}^{2n} x^i (\sum_{j=0}^{2m} x^j z x^{2m-j}) x^{2n-i} \end{aligned}$$

Moreover expanding the last summand yields

$$\begin{aligned} \textstyle\sum_{i=0}^{2n} x^i (\sum_{j=0}^{2m} x^j z x^{2m-j}) x^{2n-i} &= z x^{2m+2n} + 2x z x^{2m+2n-1} + 3x^2 z x^{2m+2n-2} \\ &\quad + \cdots + (2n+1) x^{2n} z x^{2m} \\ &\quad + \cdots + (2n+1) x^{2m} z x^{2n} \\ &\quad +2n x^{2m+1} z x^{2n-1} + \cdots + x^{2m+2n} z \end{aligned}$$

Therefore $\varphi_m\varphi_n(z) = (2n+1)\varphi_{m+n}(z)$. □

Proposition 3.4. *The Lie bracket is given by* *(i)* $[\varphi_m,\varphi_n] = 2(n-m)\varphi_{m+n}$,

(ii) $[\varphi_m,\rho_n] = (2n+1)\rho_{m+n}$,

(iii) $[\rho_m,\rho_n] = 0$.

Proof. In each case, both expressions coincide on the generators, by Lemma 3.3. Therefore they are equal as derivations. □

Remark 3.5. We observe the following facts.

(1) If we put $e_i = [\varphi_i]/2$, then the Lie algebra $\mathbb{Q} < e_1, e_2, e_3, \cdots >$ is the Witt algebra W_+. In particular it is generated by e_1 and e_2.

(2) $\mathbb{Q} < [\varphi_i] > \oplus \mathbb{Q} < [\rho_i] >$ is a $\mathbb{Z}_2$-graded algebra where $|\varphi_i| = 0$ and $|\rho_i| = 1$.

We can generalize the following:

Theorem 3.6. *If X is a product of k spheres S^{2n_i}, $i = 1,\cdots,k$, $HH^*(X^{S^1})$ contains a product of the Witt algebras $W_+\oplus\cdots\oplus W_+$.*

Proof. The product $X = S^{2n_1}\times S^{2n_2}\times\cdots\times S^{2n_k}$ admits a model $(T(V_0\oplus V_1\oplus\cdots\oplus V_{k-1}),d)$, where

$$V_0 =< x_1,\cdots,x_k >, \quad V_1 =< y_1,\cdots,y_p >,\ldots$$

such that $dV_0 = 0$, $dV_i \subset T(V_{\leq i})$. Moreover

$$H_*(TV,d) \cong H_0(TV,d) \cong T(V_0)/(dV_1).$$

Define

$$\varphi_{m,i}(x_i) = x_i^{2m+1}, \varphi_{m,i}(x_j) = 0 \text{ for } i \neq j.$$

We wish to extend inductively $\varphi_{m,i}$ on $V_1\oplus\cdots\oplus V_{k-1}$ into a derivation that is a cycle but not a boundary. Assume that $\varphi_{m,i}$ is defined on $V_0,\cdots,V_t$, $(t \geq 1)$ such that

$$[d,\varphi_{m,i}] = 0 \quad \text{on} \quad V_0,\ldots,V_t \quad \text{and} \quad \varphi_{m,i}(V_t) \subset (T(V))_{t,*}.$$

This is easily done for $t = 1$. Now take $v \in V_{t+1}$. By induction hypothesis

$$0 = [d,\varphi_{m,i}](dv) = d(\varphi_{m,i}(dv)) \in (T(V))_{t,*}.$$

As $H_*(TV,d) \cong H_0(TV,d)$, there is $v' \in (T(V))_{t+1,*}$ such that $dv' = \varphi_{m,i}(dv)$, define $\varphi_{m,i}(v) = v'$. Clearly $[d,\varphi_{m,i}] = 0$ on V_{t+1} as well.

Using a similar argument as in proof to Proposition 3.2, it is easily seen $[\varphi_{m,i},\varphi_{n,i}] = 2(n-m)\varphi_{m+n,i}$ (Proposition 3.4).

Let us show that for $i \neq j$, $[\varphi_{m,i},\varphi_{n,j}] = 0$. Observe that $[\varphi_{m,i},\varphi_{n,j}] = 0$ on V_0 and $\varphi_{m,i}(v) \subset (v)$ for $v \in V_0$. Take $v \in V_{1,*}$. As $\varphi_{m,i}$ and $\varphi_{n,j}$ are cycles therefore,

$$\begin{aligned} 0 = [d,[\varphi_{m,i},\varphi_{n,j}]](v_1) &= d([\varphi_{m,i},\varphi_{n,j}](v_1)) - [\varphi_{m,i},\varphi_{n,j}](dv_1) \\ &= d([\varphi_{m,i},\varphi_{n,j}](v_1)), \end{aligned}$$

as $dv_1 \in T(V_0)$ for $v_1 \in V_1$ and $[\varphi_{m,i}, \varphi_{n,j}](dv_1) = 0$. Since $[\varphi_{m,i}, \varphi_{n,j}](v_1) \subset T(V)_{\geq 1,*}$, there exists $w \in T(V)_{\geq 2,*}$ such that $[\varphi_{m,i}, \varphi_{n,j}](v_1) = dw$. Define a derivation α by $\alpha(V_0) = 0$ and $\alpha(v_1) = w$. It follows that $[\varphi_{m,i}, \varphi_{n,j}] = [d, \alpha]$ on $V_0 \oplus V_1$.

Extending inductively, assume that α is defined on $V_0 \oplus V_1 \oplus \cdots \oplus V_{t-1}$ such that $[\varphi_{m,i}, \varphi_{n,j}] = [d, \alpha]$. Take $v_t \in V_t$,

$$\begin{aligned} 0 = [d, [\varphi_{m,i}, \varphi_{n,j}]](v_t) &= d([\varphi_{m,i}, \varphi_{n,j}](v_t)) - [\varphi_{m,i}, \varphi_{n,j}](dv_t) \\ &= d([\varphi_{m,i}, \varphi_{n,j}](v_t)) - [d, \alpha](dv_t) \\ &= d([\varphi_{m,i}, \varphi_{n,j}](v_t)) - d\alpha(dv_t) \\ &= d([\varphi_{m,i}, \varphi_{n,j}](v_t) - \alpha(dv_t)). \end{aligned}$$

As $[\varphi_{m,i}, \varphi_{n,j}](v_t) - \alpha(dv_t) \in T(V)_{\geq t,*}$, there exists $w' \in T(V)_{\geq t+1,*}$ such that $[\varphi_{m,i}, \varphi_{n,j}](v_t) - \alpha(dv_t) = dw'$.

It comes from Proposition 3.4 that $[\varphi_{m,i}, \varphi_{n,i}] = 2(n-m)\varphi_{m+n,i}$ on V_0. From the above discussion, one can easily deduce that $[\varphi_{m,i}, \varphi_{n,i}]$ and $2(n-m)\varphi_{m+n,i}$ represent the same homology class in $\widetilde{\mathrm{Der}}\,TV$. In the same way, we put $e^i_m = [\varphi_{m,i}]/2$, $W_{+,i} = \mathbb{Q} < e^i_1, e^i_2, e^i_3, \ldots >$ is isomorphic to the Witt algebra W_+.

Consequently $HH^*(X^{S^1})$ contains a product of the Witt algebras $W_{+,1} \oplus \cdots \oplus W_{+,k}$. □

Remark 3.7. Similarly, define $\rho_{m,i}(x_i) = x_i^{2m+2}$ and $\rho_{m,i}(x_j) = 0$ for $i \neq j$. They can be extended into non zero homology classes of $\widetilde{\mathrm{Der}}\,TV$ of odd degree. The even and odd families of derivations $\varphi_{m,i} \in L_{\bar{0}}$, $\rho_{n,i} \in L_{\bar{1}}$ are compatible with the gradation

$$\begin{aligned} [L_{\bar{0}}, L_{\bar{0}}] &\subseteq L_{\bar{0}}, \qquad [L_{\bar{1}}, L_{\bar{1}}] \subseteq L_{\bar{0}}, \\ [L_{\bar{0}}, L_{\bar{1}}] &\subseteq L_{\bar{1}}, \qquad [L_{\bar{1}}, L_{\bar{0}}] \subseteq L_{\bar{1}}. \end{aligned}$$

Hence $L_{\bar{0}} \oplus L_{\bar{1}}$ is a $\mathbb{Z}_2$-graded algebra.

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