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MORE ON THE EQUIVALENCE PROPERTIES OF RADON-NIKODÝM PROPERTY TYPES AND COMPLETE CONTINUITY TYPES OF BANACH SPACES

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Abstract

We study the equivalence property of the I- and II-Λ-Radon-Nikodým property (resp. I- and II-Λ-complete continuity property) types of Banach spaces. As a by-product, we obtain a proof of the following: F. Lust-Picard's conjecture [9]: a subset Λ of a discrete abelian group is a Rosenthal set if and only if $\mathcal{C}_\Lambda(G) \not\supseteq c_0$.

1. Introduction and Preliminaries

Applications of the geometry of Banach spaces to the study of subsets of a discrete abelian group have been of particular interest for many authors [3-7, 9-11]. In [3], Dowling introduced and studied the I-, II-Λ-Radon-Nikodým property types where Λ is a subset of a countable discrete abelian

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group. Corresponding properties were also introduced and studied in [11] for the complete continuity types. Although many interesting partial results have been obtained, it is still unknown if these property types are equivalent, even if Λ is a Riesz set. For the case where Λ is a Sidon subset of an abelian discrete group, it is well known fact that all these property types are equivalent to the noncontainment of isomorphic copies of c_0. In the second section, we extend such a result to a larger class of subsets Λ which has the Godefroy-Lust-Picard lifting property. As a direct application, we obtain a proof of a result conjectured by Lust-Picard [9] which says that Λ *is a Rosenthal set if and only if* $\mathcal{C}_\Lambda$ *contains no isomorphic copies of* c_0.

For the case of the group of the integers $\mathbb{Z}$, it is also a well known fact that the type I-N-RNP type and the II-N-RNP type are equivalent properties for Banach spaces. The first proof of such an equivalence can be inferred from the result of Bukhvalov and Danielevich in [1]. The result in [10] significantly improves such a result to the more general case of discrete ordered groups. Namely, the types I and II-Λ-Radon-Nikodym property of Banach spaces are equivalent Banach space properties whenever Λ is an ordering set. And the results also applies to the I- and II-Λ-complete continuity properties. In the third section of this paper, we discuss some equivalence properties when either Λ contains or is contained in an ordering set.

Our notation and terminology are quite standard. Throughout this note, G will denote a compact connected metrizable abelian group, $\mathcal{B}(G)$ is the σ-field of Borel subsets of G, and ℓ is the normalized Haar measure on G. The dual group of G is a discrete group and will be denoted by $\hat{G}$. For a Banach space X, a measure $\mu : \mathcal{B}(G) \to X$ is said to be of *bounded variation* if the quantity

$$\| \mu \|_1 = \sup \left\| \sum_{A \in \pi} \frac{\mu(A)}{\ell(A)} 1_A \right\|_1$$

is finite, where the supremum is taken over all finite partitions π consisting of

Borel subsets of G; the measure μ is said to be of *bounded average range* if

$$\|\mu\|_\infty = \sup_{A\in\mathcal{B}(G)} \frac{\|\mu(A)\|}{\ell(A)}$$

is definite as a finite number. We denote by $\mathcal{M}^1(G, X)$ (resp. $\mathcal{M}^\infty(G, X)$) the Banach space of all X-valued countably additive measures on G that are of bounded variation (resp. of bounded average range). The subspace of $\mathcal{M}^1(G, X)$ consisting of measures that are ℓ-absolutely continuous will be denoted by $\mathcal{M}_a^1(G, X)$. The notation $L^p(G, X)$ $(1 \le p < \infty)$ (resp. $L^\infty(G, X)$) denotes the usual Banach space of all (classes of) ℓ-Bochner p-integrable (resp. essentially bounded) functions on G with values in the Banach space X. The space of all X-valued continuous functions will be denoted by $\mathcal{C}(G, X)$. The Fourier coefficient of a function $f \in L^1(G, X)$ (resp. $\mu \in \mathcal{M}^1(G, X)$) is the element of X given for each $\gamma \in \hat{G}$ by

$$\hat{f}(\gamma) = \int_G f(t)\overline{\gamma}(t)\,d\ell(t) \quad \left(\text{resp. } \hat{\mu}(\gamma) = \int_G \overline{\gamma}(t)\,d\mu(t)\right).$$

For every $\Lambda \subset \hat{G}$ and any subspace $\mathcal{S}(G, X)$ of $\mathcal{M}^1(G, X)$, the subset

$$\{\mu \in \mathcal{S}(G, X) : \hat{\mu}(\gamma) = 0 \text{ for } \gamma \notin \Lambda\}$$

will be denoted by $\mathcal{S}_\Lambda(G, X)$. It is easily verified that $L_\Lambda^p(G, X)$ is a closed subspace of $L^p(G, X)$. If $X = \mathbb{C}$, then $\mathcal{S}_\Lambda(G, X)$ will be simply denoted by $\mathcal{S}_\Lambda(G)$.

A subset $\Lambda \subset \hat{G}$ is said to be a *Riesz set* if $\mathcal{M}_a^1(G, X) = L_\Lambda^1(G)$; it is said to be *Rosenthal* if $L_\Lambda^\infty(G) = \mathcal{C}_\Lambda(G)$; it is said to be a *Sidon set* if $\mathcal{C}_\Lambda(G) \cong \ell^1(\Lambda)$. Note that Sidon sets are Rosenthal sets, and Rosenthal sets are Riesz sets.

A vector measure μ (in $\mathcal{M}^1(G, X)$) is said to be *Bochner differentiable* if there exists a function $g \in L^1(G, X)$ such that $\mu(A) = \int_A g\,d\ell$ for every

$A \in \mathcal{B}(G)$. We identify the subspace of $\mathcal{M}^1(G, X)$ (resp. $\mathcal{M}^\infty(G, X)$) consisting of differentiable measures to the function space $L^1(G, X)$ (resp. $L^\infty(G, X)$).

The following properties were first introduced by Edgar [5] and Dowling [3].

Definition 1. Let G be a compact metrizable abelian group, let $\Lambda \subset \hat{G}$, and let X be a Banach space X. Then X is said to have type:

(1) I-Λ-*Radon-Nikodým property* (I-Λ-*RNP*) if $\mathcal{M}^\infty_\Lambda(G, X) \subset L^1(G, X)$;

(2) II-Λ-*Radon-Nikodým property* (II-Λ-*RNP*) if $\mathcal{M}^1_{a,\Lambda}(G, X) \subset L^1(G, X)$.

A vector measure μ (in $\mathcal{M}^1(G, X)$) is said to have *relatively compact range* if the closure of the set $\{\mu(A) : A \in \mathcal{B}(G)\}$ is compact. The following properties were introduced in [11].

Definition 2. Let G be a compact metrizable abelian group, let $\Lambda \subset \hat{G}$, and let X be a Banach space X. Then X is said to have type:

(1) I-Λ-*complete continuity property* (I-Λ-*CCP*) if $\mathcal{M}^\infty_\Lambda(G, X) \subset \mathcal{K}(G, X)$;

(2) II-Λ-*complete continuity property* (II-Λ-*CCP*) if $\mathcal{M}^1_\Lambda(G, X) \subset \mathcal{K}(G, X)$.

One notices that of the types I-, II-$\hat{G}$-RNP coincide with the usual Radon-Nikodým property (RNP) while all of the types I-, II-$\hat{G}$-CCP coincide with the usual complete continuity property (CCP). The following implications are immediate from the definitions:

$$\begin{array}{ccccc} RNP & \Rightarrow & II-\Lambda-RNP & \Rightarrow & I-\Lambda-RNP \\ \Downarrow & & \Downarrow & & \Downarrow \\ CCP & \Rightarrow & II-\Lambda-CCP & \Rightarrow & I-\Lambda-CCP \end{array}$$

For a Sidon set Λ, we have

$$\begin{array}{ccccc} II-\Lambda-RNP & \Leftrightarrow & I-\Lambda-RNP & & \\ \Updownarrow & & \Updownarrow & & \\ II-\Lambda-CCP & \Leftrightarrow & I-\Lambda-CCP & \Leftrightarrow & \not\supseteq c_0 \end{array} \tag{1.1}$$

where the symbol $\not\supseteq c_0$ means noncontainment of isomorphic copies of c_0.

For details on the above facts, we refer the readers to [1, 3, 5] and [11].

2. Equivalence Properties and Noncontainment of Isomorphic Copies of c_0

Our first result extends diagram (1.1) to a more general class of subsets Λ of $\hat{G}$. A subset Λ of $\hat{G}$ is said to have the *Godefroy-Lust-Picard's lifting property* (ρ) [6] or a GLP set for short if there exists $\rho \in L_\Lambda^{\infty *}(G)$ such that Λ is Borel on $(L_\Lambda^{\infty *}(G), \textit{weak}^*)$, and $\rho(f) = f(0)$ for every $f \in \mathcal{C}_\Lambda(G)$. Such a subset is necessary a Riesz set [6]. We note that Rosenthal sets, and in particular Sidon sets are GLP sets. However, Li [8] showed the existence of subset $\Lambda \subset \mathbb{Z}$, which is a GLP set but fails to be Rosenthal.

Theorem 3. *Let G be a compact metrizable abelian group, and let Λ be a GLP subset of $\hat{G}$. Then*

$$\begin{array}{ccccc} II-\Lambda-RNP & \Leftrightarrow & I-\Lambda-RNP & & \\ \Updownarrow & & \Updownarrow & & \\ II-\Lambda-CCP & \Leftrightarrow & I-\Lambda-CCP & \Leftrightarrow & \not\supseteq c_0. \end{array}$$

Proof. We only need to show that if a Banach space X contains no copy of c_0, then it has type II-Λ-RNP. Suppose that Λ is a GLP set and that the Banach space X contains no isomorphic copies of c_0. Let $\mu \in \mathcal{M}_\Lambda^1(G, X)$. For each $x^* \in X^*$, the mapping

$$T_{x^*\mu} : L_\Lambda^\infty(G) \to L_\Lambda^\infty(G), \qquad T_{x^*\mu}(f) = f * x^*\mu,$$

is weak* to weak*-continuous. A result of Christensen [2] ensures the existence of a function $g_{x^*} \in L^1(G)$ such that

$$\rho(f * x^*\mu) = f * x^*\mu(0) = \int_G fg_{x^*}d\ell$$

for every *f* in $L^\infty_\Lambda(G)$. For each $\omega \in G$, we define $\rho(\mu)(\omega)$ to be the element in X^{**} such that

$$\langle x^*, \rho(\mu)(\omega)\rangle = g_{x^*}(-\omega).$$

We then consider the X^{**}-valued measure defined by

$$\nu(A) = \int_G 1_G(\omega)\rho(\mu)(\omega)d\ell(\omega),$$

for each $A \in \mathcal{B}(G)$. We notice that for each $A \in \mathcal{B}(G)$, and for each $x^* \in X^*$,

$$\langle x^*, \nu(A)\rangle = \left\langle x^*, \int_G 1_A(\omega)\rho(\mu)(\omega)d\ell(\omega)\right\rangle$$

$$= \int_G 1_A(\omega)\langle x^*\rho(\mu)(\omega)\rangle d\ell(\omega)$$

$$= \int_G 1_A(\omega)g_{x^*}(-\omega)d\ell(\omega).$$

Clearly, the mapping $x^* \mapsto \langle x^*, \nu(\cdot)\rangle$ from $X^* \to \mathcal{C}(G)^*$ is weak* to weak*-continuous. Therefore, we can define an operator $S : \mathcal{C}(G) \to X$ by

$$x^*Sf = \int_G f(\omega)g_{x^*}(-\omega)d\ell(\omega)$$

for each $f \in \mathcal{C}(G)$, and for each $x^* \in X^*$. By our hypothesis, the Banach space *X* contains no isomorphic copies of c_0, thus the operator *S* is weakly compact and the measure ν actually takes all its values in *X*.

On the other hand, we have for each $\gamma \in \hat{G}$, and for each $x^* \in X^*$,

$$x^*\hat{\mu}(\gamma) = \gamma * x^*\mu(0) = \rho(\gamma * x^*\mu)$$

$$= \int_G \gamma(\omega) g_{x^*}(-\omega) d\ell(\omega) = \langle \hat{\nu}(\gamma), x^* \rangle.$$

We conclude that $\mu = \nu$ and that $\frac{d\mu}{d\ell} = \rho(\mu)$, i.e., X has II-Λ-RNP. The proof is complete. □

Remark. F. Lust-Picard conjectured that a subset Λ is Rosenthal as soon as $\mathcal{C}_\Lambda(G) \not\supseteq c_0$. On the other hand, Li [7] asked whether or not $\mathcal{C}_\Lambda(G) \not\supseteq c_0$ is equivalent to Λ being a GLP set. In [8], Li gave a proof of the Lust-Picard conjecture under the extra condition, namely, that $\mathcal{C}_\Lambda(G)$ have unconditional metric approximation property. Theorem 3 can be used to settle all of these questions.

Proposition 4. *If Λ is a non-Rosenthal GLP set, then $\mathcal{C}_\Lambda(G)$ must contain an isomorphic copy of c_0.*

Proof. Let Λ be a non-Rosenthal GLP set, and assume to the contrary that $\mathcal{C}_\Lambda(G) \not\supseteq c_0$. Then Theorem 3 implies that $\mathcal{C}_\Lambda(G)$ must have $I - \Lambda - RNP$. It follows from [3, Proposition 5] that Λ is a Rosenthal set. Contradiction! □

This proposition shows that F. Lust-Picard's conjecture is now proved as a theorem.

Theorem 5. *A subset Λ of the dual group of a metrizable abelian group is a Rosenthal set if and only if $\mathcal{C}_\Lambda(G) \not\supseteq c_0$.*

The results of the above Theorem 3 can be slightly improved. We recall that a subset Λ of $\hat{G}$ is said to be a *Riesz set of type* 0 if $\mathcal{M}_0(G)^\wedge|_\Lambda \cong L^1(G)^\wedge|_\Lambda$, where $\mathcal{M}_0(G) = \{\mu \in \mathcal{M}(G) : \hat{\mu} \in c_0(\hat{G})\}$. Suppose that Λ_0 is a GLP set; and Λ is a subset of $\hat{G}$ that is Riesz of type 0, and that the Banach space X contains no isomorphic copies of c_0. Let $\mu \in \mathcal{M}^1_{\Lambda \cup \Lambda_0}(G, X)$. For each $x^* \in X^*$, let $h_{x^*} \in L^1(G)$ be such that $(x^*\mu)^\wedge(\gamma) = \hat{h}_{x^*}(\gamma)$ for every $\gamma \in \Lambda$. Then the measure $x^*\mu - h_{x^*} \cdot \ell \in \mathcal{M}^1_{\Lambda_0}(G)$. Applying the same arguments as in the proof of the previous theorem to the mapping

$$T_{x^*\mu - h_{x^*} \cdot \ell} : L^\infty_{\Lambda_0}(G) \to L^\infty_{\Lambda_0}(G), \quad T_{x^*\mu - h_{x^*} \cdot \ell}(f) = f * (x^*\mu - h_{x^*}),$$

we obtain

Theorem 6. *Let G be a compact metrizable abelian group, and let Λ_0 be a GLP subset of $\hat{G}$ and Λ be a Riesz subset of $\hat{G}$ of type* 0. *Then*:

$$\begin{array}{ccccc} II - \Lambda \cup \Lambda_0 - RNP & \Leftrightarrow & I - \Lambda \cup \Lambda_0 - RNP & & \\ \Updownarrow & & \Updownarrow & & \\ II - \Lambda \cup \Lambda_0 - CCP & \Leftrightarrow & I - \Lambda \cup \Lambda_0 - CCP & \Leftrightarrow & \not\supseteq c_0. \end{array}$$

3. More Equivalence Properties

In [10], it is shown that if Λ is an ordering subset of $\hat{G}$, that is, if Λ satisfies $\Lambda + \Lambda \subset \Lambda$, $\Lambda \cap (-\Lambda) = \{0\}$, and $\Lambda \cup (-\Lambda) = \hat{G}$, then we have

$$\begin{array}{ccc} II - \Lambda - RNP & \Leftrightarrow & I - \Lambda - RNP \\ \Downarrow & & \Downarrow \\ II - \Lambda - CCP & \Leftrightarrow & I - \Lambda - CCP. \end{array}$$

In fact, the proofs of such results can actually be used to prove the following:

Theorem 7. *Let G be a compact connected metrizable abelian group and let Λ be a subset of the dual group $\hat{G}$ that contains an ordering subset Λ_0 of $\hat{G}$. Then*

$$\begin{array}{ccc} II - \Lambda - RNP & \Leftrightarrow & I - \Lambda - RNP \\ \Downarrow & & \Downarrow \\ II - \Lambda - CCP & \Leftrightarrow & I - \Lambda - CCP. \end{array}$$

For the proof, we denote by Π the set of all trigonometric polynomials P

on G of the form $P = \sum_{i=1}^{n} a_{\gamma_i}\gamma_i$, where $\gamma_i \in \Lambda_0$. Let $w \in L^1(G)$, and $w \geq 0$. Then consider the closure K of the set $\{1 + P : P \in \Pi\}$ in the Hilbert space $L^2(G, wd\ell)$. Since K is convex, there is a unique $\varphi \in K$ such that

$$\| \varphi \|_{L^2(G, wd\ell)} = \inf_{P \in \Pi} \| 1 + P \|_{L^2(G, wd\ell)}.$$

The following lemma is needed the proof of which can be inferred from [12, p. 199].

Lemma 8. *In the above situation*, φ *has the following properties*:

(1) $\varphi w \in L^2(G)$ *and* $|\varphi|^2 w = \| \varphi \|_{L^2(G, wd\ell)}\ell$ *-almost everywhere*;

(2) *if* $\| \varphi \|_{L^2(G, wd\ell)} > 0$, *then* $1/\varphi \in L^2_\Lambda$ *and* $\widehat{1/\varphi}(0) = 1$.

The arguments in the proof of Theorem 7 are essentially the same as in the proof of [10, Theorems 1 and 2].

Proof. Assume first that X is a Banach space with the type I-Λ-RNP, and $\mu \in \mathcal{M}^1_{\Lambda_0}(G, X)$. Let φ be the unique function of Lemma 8 associated to the function $w = \frac{d|\mu|}{d\ell}$. Now consider the measure $d\nu = \varphi^2 d\mu$. Then, for every $A \in \mathcal{B}(G)$, we have

$$\begin{aligned} \| \nu(A) \|_X &= \left\| \int_A \varphi^2 d\mu \right\|_X \leq \int_A |\varphi|^2 d|\mu| \\ &\leq \int_A |\varphi|^2 wd\ell = \| \varphi \|_{L^2(G, |\mu|)}\ell(A). \end{aligned}$$

That is, the measure $\nu \in M^\infty(G, X)$. We claim that $\hat{\nu}$ vanishes outside Λ.

To see this, we first prove that the measure σ defined by $\varphi d\mu$ is such that $\hat{\sigma}(\gamma) = 0$ if $\gamma \notin \Lambda$. Fix a sequence (P_n) in σ such that

$$\lim_{n \to \infty} \int_G |\varphi - (1 + P_n)|^2 d|\mu| = 0.$$

Now, for each $\gamma \in \hat{G},$ we have by the Cauchy-Schwarz inequality

$$\begin{aligned}\| \hat{\sigma}(\gamma) - ((1 + P_n)d\mu)^\wedge(\gamma) \| &= \left\| \int_G [\bar{\gamma}\varphi - \bar{\gamma}(1 + P_n)] d\mu \right\| \\ &\le \int_G | \varphi - (1 + P_n) | d|\mu| \\ &\le |\mu|(G)^{1/2} \left(\int_G | \varphi - (1 + P_n)|^2 d|\mu| \right)^{1/2}.\end{aligned}$$

This inequality holds for all n, therefore,

$$\hat{\sigma}(\gamma) = \lim_{n \to \infty} ((1 + P_n)d\mu)^\wedge(\gamma).$$

Since $(1 + P_n)d\mu \in \mathcal{M}_\Lambda^1(G, X)$ for all $n \in N,$ we have $\sigma \in \mathcal{M}_\Lambda^1(G, X).$

Repeating the same argument, we have for each $\gamma \in \hat{G},$

$$\begin{aligned}&\| \hat{\nu}(\gamma) - ((1 + P_n)\varphi d\mu)^\wedge(\gamma) \| \\ &= \left\| \int_G [\bar{\gamma}\varphi^2 - \bar{\gamma}(1 + P_n)\varphi] d\mu \right\| \\ &\le \int_G | \varphi^2 - (1 + P_n)\varphi | d|\mu| \\ &= \int_G |\varphi| |\varphi - (1 + P_n)| d|\mu| \\ &\le \| \varphi \|_{L^2(G, |\mu|)} \left(\int_G | \varphi - (1 + P_n)|^2 d|\mu| \right)^{1/2}.\end{aligned}$$

Hence, we also obtain

$$\hat{\nu}(\gamma) = \lim_{n \to \infty} ((1 + P_n)\varphi d\mu)^\wedge(\gamma).$$

Since $\varphi d\mu \in \mathcal{M}_\Lambda^1(G, X),$ the measure $(1 + P_n)\varphi d\mu \in \mathcal{M}_\Lambda^1(G, X)$ for all $n \in N,$ we have $\nu \in \mathcal{M}_\Lambda^1(G, X).$ This proves our claim.

Hence $\nu \in \mathcal{M}_{\Lambda}^{\infty}(G, X)$, and therefore $\frac{d\nu}{d\ell}$ exists in $L^1(G, X)$ and we have for each $A \in \mathcal{B}(G)$,

$$\mu(A) = \int_A \frac{1}{\varphi^2} \varphi^2 d\mu = \int_A \frac{1}{\varphi^2} d\nu = \int_A \frac{1}{\varphi^2} \frac{d\nu}{d\ell} d\ell.$$

This shows that the measure μ is differentiable and $\frac{d\mu}{d\ell} = \frac{1}{\varphi^2} \frac{d\nu}{d\ell}$.

Assume now that X has the type I-Λ-CCP, and let $\mu \in \mathcal{M}_{a,\Lambda}^{1}(G, X)$. We wish to show that Λ has relatively compact range. It is enough to show that the operator $T : L^{\infty}(G) \to X$ defined by $T(f) = \int_G f d\mu$ is compact. By our above claim $d\nu = \varphi^2 d\mu$ belongs to $\mathcal{M}_{\Lambda}^{\infty}(G, X)$. Thus, by our hypothesis on X, the operator $S : L^1(G) \to X$ defined by $Sf = \int_G f d\nu$ is completely continuous.

Now let (f_n) be bounded in $L^{\infty}(G)$. By passing to a subsequence if necessary, we can assume that the sequence (f_n) converges weak* to some function $f \in L^{\infty}(G)$. It then follows that the sequence $((f_n - f)\varphi^{-2})$ is weakly null in $L^1(G)$, and hence the $(S((f_n - f)\varphi^{-2}))$ is norm null in the Banach space X. To finish the proof, we notice that

$$\begin{aligned} S((f_n - f)\varphi^{-2}) &= \int (f_n - f)\varphi^{-2} d\nu = \int (f_n - f)\varphi^{-2}\varphi^2 d\mu \\ &= \int (f_n - f)\varphi^{-2}\varphi^2 d\mu = \int (f_n - f) d\mu = T(f_n - f). \end{aligned}$$

This shows that $T(f_n - f)$ is norm null in X. The proof is complete. □

We also notice the following:

Proposition 9. *Let G be a compact connected metrizable abelian group and let* Λ_0 *be an ordering subset of the dual group* $\hat{G}$*. If a Banach space X* *has type* $I - \Lambda_0 - RNP$ $\left(resp.\ I - \Lambda_0 - CCP\right)$, *then it has* $II - \Lambda - RNP$ $\left(resp.\ II - \Lambda - CCP\right)$ *for any subset* Λ *of* Λ_0.

Proof. Note that, since X has $I - \Lambda_0 - RNP$, it has $II - \Lambda_0 - RNP$. Hence, if $\mu \in \mathcal{M}^{\infty}_{\Lambda}(G, X)$, then $\mu \in \mathcal{M}^{1}_{\Lambda_0}(G, X)$, and therefore, μ is Bochner differentiable (resp. μ has relatively compact range). □

As a consequence of Theorem 7 and Proposition 9, we have:

Corollary 10. *Let G be a compact connected metrizable abelian group and let* Λ *be a subset of the dual group* $\hat{G}$. *Then*

$$\begin{array}{ccc} II - \Lambda - RNP & \Leftrightarrow & I - \Lambda - RNP \\ \Downarrow & & \Downarrow \\ II - \Lambda - CCP & \Leftrightarrow & I - \Lambda - CCP \end{array}$$

for Banach spaces with type $I - \Lambda_0 - RNP$, *where* Λ_0 *is an ordering subset of* $\hat{G}$.

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